THE CONTRIBUTION OF VEDIC MATHEMATICS IN ADVANCE CALCULUS

A

THESIS

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By

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I declare that this thesis entitled **“THE CONTRIBUTION OF VEDIC MATHEMATICS IN ADVANCE CALCULUS”** is my own work conducted under the supervision of Dr. Prabha S. Rastogi, Supervisor, Department of Mathematics at Shri Jagdishprasad Jhabarmal Tibrewala University, Vidyanagari, Jhunjhunu, Rajasthan, approved by the Research Degree Committee of the University. I also declare that I have put in more than 200 days/600 hours of attendance with the supervisor.

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ABSTRACT

Vedic mathematics was rediscovered from the ancient Indian scriptures between 1911 and 1918 by SWĀMĪ ŚRĪ BHĀRATĪ KṚṢṆ A TĪRTHAJĪ MAHĀRĀJA Śankarācārya of Govardhan Matha, Puri (1884-1960) through intuitive revelation.

Today, amazing subfield of mathematics-Vedic Mathematics is gradually making a comeback and establishing a stronghold in the modern education system of the entire world.

In this thesis an effort has been made to illuminate the research topic **"The Contribution of Vedic Mathematics in Advance Calculus"** by explaining almost each and every Sūtras with meaning, their applications in solving all types of equations starting from linear, quadratic, cubic, quartic to ordinary and partial; linear and non-linear differential equations. Also determinant and matrices; in differential calculus for finding derivative of product of two and three functions, successive differentiation till nth order derivative of the product of two functions by using Leibniz, Taylor's and Maclaurin's theorem; integration problems based on partial fraction and integration by parts are also demonstrated. The thesis explores solving problems by using Sūtras not only in a wide area of mathematics but also in computer arithmetic. Vedic mathematics is useful for calculating binary operations like addition, subtraction, multiplication and division and also binary squaring and binary cubing. The researcher has tried to elucidate the application of the Sūtras by applying them to a variety of new and original problems and newer and unique areas. The Sūtras and Sub-Sūtras are remembered in the form of word formulae, and once we understand the meaning and the pattern of it, one can apply corresponding Sūtras in solving the problem in few seconds without writing so many steps in between like conventional mathematics. The age of the learner and qualifications do not pose a problem in this process and very little effort is required to break the dependence on calculators. In short, Vedic Mathematics is an aid to enhance calculation skill.

Researcher has concluded that the most important feature of Vedic mathematics is its consistency. Because of this quality it makes mathematics easy and enjoyable. It also inspires innovations.

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LIST OF VEDIC SŪTRAS:

Sr. No.	Title of Vedic Sūtras	Meaning of Vedic Sūtras

1.	Ekādhikena Pūrveṇa Sūtra	By one more than the previous one
2.	Nikhilam Navataścaram am Daśtaḥ Sūtra	All from nine and the last from 10
3.	Ūrdhva-Tiryagbhyām Sūtra	Vertically and Crosswise
4.	Parāvartya Yojayet Sūtra	Transpose and apply
5.	Śunyam Samyasamuccaye Sūtra	If the samuccaya is the same that samuccaya is zero.
6.	(Ānurūpye) Śunyamanyat Sūtra	If one is in the ratio the other is zero.
7.	Saṅkalana-Vyavaalanābhyām Sūtra	By addition and Subtraction
8.	Pūraṇāpūrṇābhyām Sūtra	By completion or non-completion
9.	Calana-kalanābhyām Sūtra	Differential Calculus
10.	Yāvadūnam Sūtra	By the Deficiency
11.	Vyaṣṭisamṣtiḥ Sūtra	Whole as one and one as whole
12.	Śeṣāṇyaṅkena Carameṇa Sūtra	The remainders by the last digit
13.	Sopāntyadvayamantyam Sūtra	The ultimate and twice the penultimate
14.	Ekanyūnena Pūrveṇa Sūtra	By one less than one before
15.	Guṇitasamuccaayaḥ Sūtra	The product of the sum
16.	Guṇaksamuccayaḥ Sūtra	All the multipliers

LIST OF VEDIC SUB-SŪTRAS:

Sr. No	Title/ Description of vedic Sub-Sūtras	Meaning of Vedic Sub-Sūtras
1.	Ānūrūpyeṇa Sub-Sūtra	Proportionality
2.	Śiṣyate Śeṣasam jñaḥ Sub-Sūtra	The remainder remains Constant

3.	Ādyamadyenāntyamantyena Sub-Sūtra	First by the first and last by the last
4.	Kevalaiḥ Saptkam Gunyāt Sub-Sūtra	For 7 the multiplicand is 143
5.	Veṣṭnam Sub-Sūtra	Osculators
6.	Yāvadūnam□ Tāvadūnam□ Sub-Sūtra	Whatever the extent of its deficiency
7.	Yāvadūnam□ Tāvadūnīkr□tya Vargañca Yojyet Sub-Sūtra	Deviate as much as deviation and add square of deviation
8.	Antyayordaśake'pi Sub-Sūtra	The total of last digit is 10 and the previous part is exactly same.
9.	Antyayoreva Sub-Sūtra	only the last term
10.	Samuccayguṇitaḥ Sub-Sūtra	The sum of the co-efficient is the product along with Guṇitasamuccayaḥ Samuccayaguṇitah
11.	Lopanasthapanābhyām Sub-Sūtra	by alternate elimination and retention
12.	Viloknam Sub-Sūtra	By observation
13.	Guṇitasmuccayaḥ Samuccayaguṇitaḥ Sub-Sūtra	The product of the sum of the coefficients in the factor is equal to the sum of the coefficients in the product.

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Chapter-1

Introduction and Mathematical Prerequisites

1.1 Introduction to research:

Eminent Indian Mathematicians' contribution to mathematics is truly immense, they gave us the value of zero, the use of decimals, complex mathematical calculations etc. From the traditional period till the eighteen century, contributions by Indian mathematicians have been exceptionally abundant in which leading scholars like Aryabhatta, Bhaskaracharya, Brahmgupta and many others have given big inputs to the field of mathematics. Along with all these ancient mathematicians of India, there also developed Vedic Mathematics which is based on **Sūtras** and **Sub-Sūtras** the form of word formulae written in Sanskrit. This research focuses on Vedic Mathematics especially with relation to its application in advance Calculus.

1.2 Research Problem at hand:

Although Vedic mathematics was found in India, but Indians have not shown much interest in Vedic Maths. On the other hand, the western world is embracing Vedic Mathematics in a big way and is being charmed by the secrets of Vedic Maths. Today, Vedic Mathematics is gradually making strong come back in the education system. Today, a lot of research is being done on the comparison of Vedic Mathematics with regular mathematics. Large number of researchers have found many advantages of Vedic Mathematics over regular mathematics. They have shown that sometimes, solving problem by using regular mathematical steps, is more time consuming and complex in comparison to Vedic Mathematics. Some researchers have elucidated that one can solve the problems of Mathematics 10-15 times faster than usual method by using Vedic Mathematics Sūtras. The beginning of Vedic Mathematics is surrounded by mystery and there is no clarity about the origins of Vedic mathematics. While there is a lot of confusion and debate about the origins of Vedic mathematics, this research does not concern itself with this debate. The current research tries to find out whether the techniques of Vedic mathematics which are useful in performing basic mathematical operations like multiplication, division, squaring and cubing are also helpful in solving higher level mathematical problems. The current research is an attempt to find out the extent to which Vedic Sūtras can be helpful in solving quadratic equations, cubic equations, quartic equations faster than conventional method; finding determinant and in finding solutions to the problems of matrices like finding adjoint of matrix, inverse of matrix and rank of a matrix. It is an attempt to comprehend and solve the problems on derivatives, integrations and differential equations by using Vedic Sūtras. In computer arithmetic as well, it tries to look how Vedic mathematics helps in calculating binary tasks like sum, difference and product and furthermore binary squaring and binary cubing.

Aim of Research:

The researcher's objective is to find out the Contribution of Vedic Mathematics in Advance Calculus.

Objectives of Research:

To apply the Vedic mathematics Sūtras to classical mathematics problems and comparing its utility with that of conventional algebraic proof.

To analyze the applications of Vedic mathematics Sūtras in determinant, matrices and finding determinant of Jacobian matrix; solving special types of quadratic equations and factorization of quadratic, cubic and bi-quadratic equations; differentiation, successive differentiations covering Taylor's series and Maclaurin's series, Integration as well as ordinary and partial differentiation.

To formulate algorithms using Vedic Mathematics Sūtras.

1.3 Need and Importance of the Research:

The Sūtras and Sub-Sūtras of Vedic mathematics are applicable to all the fields of Mathematics which includes areas like Calculus- Differential and Integral, Algebra, Arithmetic, Astronomy, Geometry, Trigonometry etc. The formulae of Vedic mathematics are applicable to complex and lengthy problems which required many mathematical operations. Application of Sūtras enhances the calculation ability of the pupils in large extent of complexities, which ensures speedier & accurate calculations related to logical & rational reasoning. By using Vedic Mathematics, solutions to complex mathematical questions are obtained within two-three lines and also sometimes orally without writing anything. Therefore, with the knowledge of Vedic method and its applications to the particular topic one can achieve their requirement to solve and complete the paper within the time limit and clear the challenging exams like GRE, Graduate Management Admission Test, Union Public Service Commission Exam, Common Admission Test for MBA, Common Entrance Test for the admission in the Medical and Engineering Exam. As many calculations are possible to be done in the mind itself, Vedic maths is also known as 'Mental Maths' and because of this it improves the memory of the students and helps them to enjoy studies. In short, Vedic Mathematics offers single step, intellectual and high-speed methods while providing methods for the verification of the result. According to researcher, Vedic maths contains the top-secret of quick calculations and solving mathematical problems intellectually. So there is a need to replace outdated conventional methods.

1.4 Methodology:

This research does not contain any statistical method like t-test, z-test and Chi square- test. It will be based on analysis and review of different literature based on Vedic Mathematics Sūtras related to higher mathematics and advance calculus.

Solving certain unique quadratic equations more effortlessly and more swiftly with the help of the Sūtras like Viloknam Sub-Sūtra, Parāvartya Yojayet, Śūnyam□ Sāmyasamuccaye Sūtra, Ānurūpye Śūnyamanyat Sūtra.

Researcher will apply Ādyamadyenāntyamantyena and Ānurūpye□a Sub-Sūtra for factorization of quadratic equation in two variables; Ādyamadyenāntyamantyena & Lopanasthāpanābhyām Sub-Sūtras for factorization of long homogeneous equations with three variables; and independent term along with three as well as four variables; Also for finding 1st, 2nd and 3rd order derivatives of product of two and three functions, even nth order derivative of product of two functions by Vertically & Crosswise. And Gu□akasamuccaya□ Sūtra is helpful for finding successive differentiation, which covers theorems like Maclaurin’s and Taylor’s and compares the mathematical truth of the well-known rule of differentiation and Gu□akasamuccaya□ Sūtra, process of Integration with the help of using Ekādhikena Pūrve□a and Vertically & Crosswise Sūtra, problems based on Partial Divisions by Parāvartya Yojayet Sūtra; determinant as well as matrix and Jacobian matrix by Vertically & Crosswise Sūtra, computer arithmetic operations for binary researcher will apply Vertically & Crosswise Sūtra, Nikhila□ Sūtra, Anurupyena, Parāvartya Yojyet Sūtra, Duplex property, Dhwaja□ka method & Yāvadūnam Tāvadunī Kr□tya Vargañca Yojayet.

1.5 Historical Overview:

The Indian civilization is one of the oldest in the world. The earlier knowledge was not codified and passed on orally for centuries before being put in a written script. The Indian sages passed on their collected works verbally from one generation to the other using codes which unlock various layers of meanings. The verbalized knowledge was accumulated into four texts in Sanskrit known as the Vedas which means knowledge. Out of four Vedas, Rig Veda is the oldest of the Vedas and contains hymns dedicated to nature. Yajur Veda and Sam Veda also contain the chants celebrating the Gods of nature while Atharva Veda contains a collection of spells, incantations and Speculative chants.

1.5.1 The Origin of Vedic Mathematics:

Mathematics and Astronomy share a profound relationship in the Indian tradition. A lot of work and derivations present in Architecture and Astronomy in the Vedas would not have been possible without the present knowledge of Mathematics. Vedas are very old and are written in cryptic language. One of the Vedas, Sthapatya Veda is full of Architecture knowledge and Vedang Jyotish an important part of Vedas deals in Astrology and Astronomy which require depth of knowledge in Mathematics. Therefore, we can say that origination of Vedic Mathematics must have come from Vedas only but decodifying Sūtras from Vedas was a difficult job. It was a tedious job to find the Vedic mathematics Sūtras in the present form.

In between the years 1911 and 1918, SWĀMĪ ŚRĪ BHĀRATĪ KR□□□A TĪRTHAJĪ MĀHĀRĀJA, re-found Vedic mathematics from ancient Indian scriptures through intuitive revelation. Vedic Maths is constructed by using sixteen Vedic Sūtras or aphorisms, and thirteen Sub-Sūtras from Atharva Veda which are actually reconstructed on the basis of spontaneous disclosure in the form of word formulas which describes regular methods to solve any kind of mathematical complexities. The sixteen Sūtras which are original thoughts, spoken or written in a short and memorable form. Although Sūtras are said to have come naturally to TĪRTHAJĪ, Sūtras & Sub - Sūtras have principles of logic behind them.

1.5.2 The founder of Vedic Mathematics:

The founder of Vedic Mathematics, SWĀMĪ ŚRĪ BHĀRATĪ KR□□□A TĪRTHAJĪ MĀHĀRĀJA was born in 1884.TĪRTHĀJĪ, called Venkatram in his younger days, was an extremely intelligent scholar. He topped his matriculate exam which was conducted by Madras University in Jan, 1899. Jagadguruji was a multi-dimensional and multi – faceted person having in-depth knowledge of Mathematics, History and Philosophy and was an erudite scholar of Sanskrit. On securing the topmost marks in the B.A. examination, he went on to do his M.A. from the American College of Sciences, New York. He appeared for his examination in the Bombay Centre in the years 1903-1904. He cleared his Master’s Degree at the age of only 20 and completed his post-graduation by studying seven subjects while simultaneously securing the highest marks in each one. This was indeed an all-time world-record of academic brilliance.

BHĀRATĪ KR□□□A TĪRTHAJĪ, was the earlier Śankarāchārya - a key spiritual head of Puri, in India, and after researching on the old texts of the Vedas, he came up with the principles of Vedic mathematics which was published in ground-breaking work Vedic Mathematics in 1965. It is the foundation of Vedic Maths. In the last part of 1960s, a print of the book Vedic Mathematics arrived in London and it drew immediate attention as a novel substitute of Maths. Some leading British Mathematicians at that time like Pickles Jeremy, Andrew Nicholas and Kenneth Williams took interest in this new approach. They modified the preliminary work of Tirthaji’s book & gave lectures on Vedic Maths at London. These talks were assimilated to form a book with the title Introductory lectures on Vedic Mathematics in the year 1981. Some sequential visits to India -by Andrew Nicholas (1981 to 1987), there was a regeneration of interest in Vedic maths among researchers and educators in India. They began taking Vedic maths more sincerely.

Figure: 1.1

SWĀMĪ ŚRĪ BHĀRATĪ KṚṢṆA TĪRTHAJĪ MĀHĀRĀJA,
(1884-1960)

Image Source:
http://www.vedicmaths.org/Community/whoswho/Tirthajiportrait.gifage

1.6 Vedic Sūtras and Comparison of Vedic Methods with Algebraic Algorithms:

Ancient Indian Vedic Mathematics is founded on 16 Sūtras and 13 Sub-Sūtras. The efficient & systematic approach towards mathematical computation of the applications of Sūtras is unchallenged. Here, the researcher has attempted to explain the different methods by using Sūtras & Sub-Sūtras and has tried to show their connectivity with Algebra, Arithmetic and Calculus as well.

Ekādhikena Pūrveṇa Sūtra:

Meaning: By one more than the previous one.

For finding Square of numbers having unit digit 5:

To find out square of a number 95 by conventional method, the result is obtained by multiplying 95 with 95.

Using Vedic Ekādhikena Pūrveṇa Sūtra squaring a number containing unit digit 5 gives one line answer.

E.g.

[1]

Find square of 95 i.e. $(95)^2$.

The digit in the unit's place is 5 Vedic Ekādhikena Pūrveṇa Sūtra is applied.

Procedure:

L.H.S. of the final answer is obtained by multiplying the first digit 9 by 10 (one more than 9).

Therefore, L.H.S. of the final answer is $9 \times 10 = 90$ and

The final answer of Right hand side is obtained by taking square of last index digit 5 i.e. $5^2 = 25$.

Concatenate the Left & Right side of the part to get final answer.

The final answer is $(95)^2 = 90 \mid 25 = 9025$

[2]

Find the square of 125.

$(125)^2 = 12 \times (12 + 1) \mid 5^2$

$= 12 \times 13 \mid 25 = 156 \mid 25 = 15625$

For conversion of vulgar fraction into Recurring Decimal:

Special division whose divisor is ending in 1 like 1/21, 1/41………and special division whose divisor is ending with 9 like 1 divided by 19, 29, 39 ….just in one step.

Denominator ending in 1:

If divisor contains unit digit 1 then by dropping 1, reduce the dividend by one. The part of the procedure which is different than the divisor ending in 9 is that after completion of the first division we prefix the remainder to the complement of quotient digit from nine not to each quotient digit and continue division in the same method.

Denominator Ending in 9:

E.g.

Convert 1/19 into equivalent vulgar fraction:

If the divisor is ending with 9 and the previous digit is 1, by the Vedic Sūtra one more than the previous one, we get 1 + 1 = 2. To covert the vulgar fraction (A fraction consisting of an integer numerator and a non-zero denominator) into recurring Decimal Ekādhikena Pūrveṇa Sūtra is used either for division (dividing by 2) or for multiplication (multiplied by 2).

Here, the difference between the numerator and denominator is 19 - 1 = 18.

Therefore, in the final answer we get 18 decimal places before repetition.

Division Method**:**

Starting with the first digit of the dividend i.e. 1

[1] $1/2 = 1^0$
[2] $10/2 = 0^5$
[3] $05/2 = 1^2$
[4] $12/2 = 0^6$
[5] $06/2 = 0^3$
[6] $03/2 = 1^1$
[7] $11/2 = 1^5$
[8] $15/2 = 1^7$
[9] $17/2 = 1^8$
[10] $18/2 = 0^9$………..

Figure: 1.2

Continuing this way till 18 steps. After that (i.e. step 19) onwards there is a repetition of step 2 to step 18. Also if we subtract the obtained result of step 1 i.e. 0 from 9 we get the result 9 which is the result of step 10. Thus, if we know answer of the first nine digit after the decimal point then remaining nine digit of the answer after that 9 digits can be obtained by subtracting answer of first nine digits from 9.

Thus, once we know the procedure the answer in one–line is

$\therefore\ 1/19 = 0.{}_{1}0\ {}_{0}5\ {}_{1}2\ {}_{0}6\ {}_{0}3{}_{1}1\ {}_{1}5\ {}_{1}7\ {}_{1}8\ {}_{0}9\ {}_{1}4\ {}_{0}7\ {}_{1}3\ {}_{1}6\ {}_{0}8\ {}_{0}4\ {}_{0}2\ {}_{0}1$

Thus, the final solution of the fraction is 0.052631578947368421

Multiplication Method:

Convert 1/19 into recurring Decimal.

For solving the above problem by multiplication method by using the Ekādhikena Pūrveṇa Sūtra starting with 1 multiplied by 2 & write it to left of 1 and continue this process till 18 steps.

1. Start with 1
   (So that the product of multiplicand and multiplier concerned may end in 9)
2. 1 × 2 = 2.Write 2 to the left side of 1 i.e. 21
3. 2 × 2 = 4. Write 4 to the left side of 21 i.e. 421
   4 × 2 = 8. Write 8 to the left side of 421 i.e. 8421
4. 8 × 2 = 16. Write 6 to the left of 8421 and 1 carried over. i.e. ${}_{1}68421$
5. 6 × 2 = 12 + 1 carried over =13.
   Write 3 to the left side of 68421 and 1 carried over.i.e.${}_{1}368421$
6. 3 × 2 = 6 +1(carried over) = 7 i.e.7368421
7. 7 × 2 = 14. i.e.${}_{1}47368421$
8. 4 × 2 = 8 +1(carried over) = 9 i.e. 947368421
9. 9 × 2 = 18 i.e. ${}_{1}8947368421$…………….

Continuing this process till the 18 steps. After 18 steps continue the process by repetition using the same number in the same order by writing it one by one to the left side. Also if we subtract the obtained result in step1. I. e. 1 from 9 we get 9 - 1 = 8 which is the result of 10th step. Thus if the answer of last 9 digits after decimal point are known then the answer of first nine digits after decimal point is obtained by subtracting the answer of last 9 digits from 9. Thus, once we know the procedure by doing mental steps the final answer in 18 digits after the decimal point is obtained in one line.

$\therefore\ 1/19 = 0.{}_{1}0\ {}_{0}5\ {}_{1}2\ {}_{0}6\ {}_{0}3{}_{1}1\ {}_{1}5\ {}_{1}7\ {}_{1}8\ {}_{0}9\ {}_{1}4\ {}_{0}7\ {}_{1}3\ {}_{1}6\ {}_{0}8\ {}_{0}4\ {}_{0}2\ {}_{0}1$

The above alternate methodology attains accurate result and verifies the solution by Division method.

E.g.

1/41 Here, the auxiliary fraction of 1/41 is

$$\therefore\ \frac{1}{41} = .\ {}_{0}0\ {}_{1}2\ {}_{1}4\ {}_{3}3\ {}_{0}9\ {}_{0}0\ {}_{1}2\ {}_{1}4\ {}_{1}3\ {}_{0}9 = 0.\dot{0}243\dot{9}$$

$$\frac{1}{39} = 0.{}_{1}0\ {}_{2}2\ {}_{2}5\ {}_{1}6\ {}_{0}4\ {}_{0}1\ {}_{1}0\ {}_{2}2\ {}_{2}5\ {}_{1}6\ {}_{0}4\ {}_{0}1$$

$$\therefore\ \frac{1}{39} = 0.02564125641\ldots\ldots\ldots = 0.\dot{0}2564\dot{1}$$

Integration of power of x:

By adding one to Pūrva (given) according to Ekādhikena Pūrvena Sūtra one can find out the integral power of x.

ILLUSTRATION:

[1] $\int 4x^3 dx = 4.\frac{x^4}{4} + c = x^4 + c$

[2] $\int (x^4 + 8x^3 + 9x^2 + 4x - 2)\, dx$

$$= \frac{x^5}{5} + 8\frac{x^4}{4} + 9\frac{x^3}{3} + 4\frac{x^2}{2} - 2x + c$$

$$= \frac{x^5}{5} + 2x^4 + 3x^3 + 2x^2 - 2x + c$$

Nikhilaṃ Navataścaramaṃ Dasatah Sūtra:

It means that subtract all numbers from nine & last number from ten.

Subtraction:

Special type of subtraction from 100, 1000 and so on.

E.g.:

Subtract 456 from 10000. i.e.10000 – 456.

First convert 456 in equivalent to the number of zero included in the 10000.

Here, four zero included in 10000.

Therefore, by converting 456 into four digit number 0456.

Then by using Nikhila□ Sūtra (all from 9 and last from 10) for 0456.

I. e. 10 - 6 = 4; 9 - 5 = 4; 9 - 4 = 5; 9 - 0 = 9, we get answer 9544.

Thus, 10000 - 456 = 9544.

Multiplication:

When we want to multiply two larger numbers which are more close to bases in multiple of 10. Three possible cases are:

Both the numbers below the base

Both the numbers are more than base number.

One number is more than base number & another number is less than base number. The given number is either add to the base or subtract from the base. Deviation (difference between number and base) may be positive or negative. If the given number is below the base deviation is negative and if the given number is above the base deviation is positive.

All the three types mentioned above are explained by taking basic problem.

[1] 14 × 17

Here, Base is 10.

Also both the numbers are higher than the base 10.

14 +4
× 17 +7
14 +7 or 17 + 4 / 7 × 4
21 / 28
238
∴ 14 × 17 = 238

[2] 97× 94

Base is 100

Also, multiplier & multiplicand both are less than base 100.

97 -03
× 94 -06

97 – 06 or 94 – 03 / 3 × 6

91 / 18

9118

Thus, 97 × 94 = 9118

If out of two nos. one is more than base number & another is less than base number then multiplication of deviation is negative.

To remove the negative part of the right side, multiply the left side by base number and subtract the R.H.S. part from it.

Concatenate right & left part to get the final result.

[3]102 × 97

Here, nearest base is 100.

Also, Multiplicand is more than base and multiplier is less than base.

102 (102 -100) = 02

× 97 (97- 100) = - 03

(102 - 3) or (97 +2) | - 06

99 | - 06

9900 - 06 = 9894

Thus, 102 × 97 = 9894

Division**:**

Divisor is closer to power of 10 but less than that of it. E.g. 92, 98, 995, 8997 etc. Deficiency write below the divisor.

Dividend is divided in two parts Quotient and Remainder has same number of digit as that of divisor. Remainder cannot be greater than or equal to the divisor.

So continue the process of division till we get remainder less than the divisor.

[1] 321 ÷ 9

9 32 / 1

1 3 / 5

35/ 6

$\therefore$ Q = 35 and R = 6

[2] 23455 ÷ 9

$\underline{9} \quad 2\ 3\ 4\ 5 \quad /\ 5$

$\underline{\ \ 1 \quad\ \ 2\ 5\ 9 \ \ /\ 14}$

$25_1 9_1 4\ / \ \ 19$

$2\ 6\ 0\ 4\ /2/1$

$2\ 6\ 0\ 6\ /\ 1$

$\therefore$ Q = 2606 and R = 1

The above procedures for multiplication and division of decimal numbers by Nikhila□ Sūtra are also applicable to binary multiplication and binary division respectively.

Ūrdhva-Tiryagbhyām Sūtra:

Meaning: By Vertically & Crosswise

The above Sūtra is very important for of all types of multiplication & for dividing big quantity by other big quantity for the decimal number as well as binary number.

It is also used in solving problems of Determinant and Matrices, in solving simultaneous linear equations, to check consistency of linear equations, for finding co-factor of the matrix, Adjoint of Matrix and also for finding Inverse of Matrix, rank of the matrix and determinant of Jacobian matrix.

For finding 1st, 2nd and 3rd order derivative of product of two functions as well as product of three functions.

For solving problems based on Integration of product of two functions.

Few apllications of the above sutra are explained by giving examples of each.

With the help of following diagrams in which each dot denotes a digit and the line represents the multiplication of digit pairs it can be remembered easily.

Line diagram of 2×2 digits, 3 × 3 digits and 4×4 digits:

For 2×2 Digits:

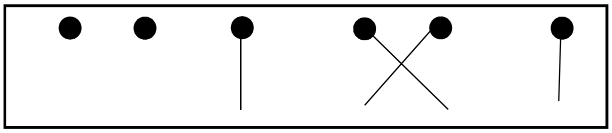


Figure-1.3

For 3×3 Digits:

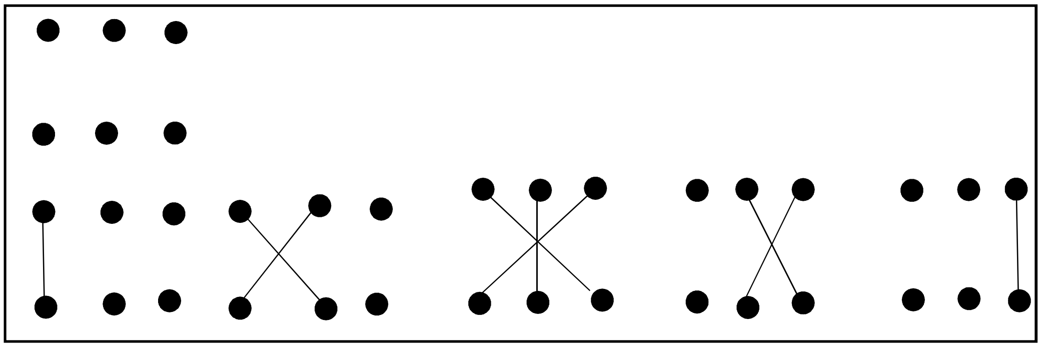


Figure 1.4

For 4×4 Digits:

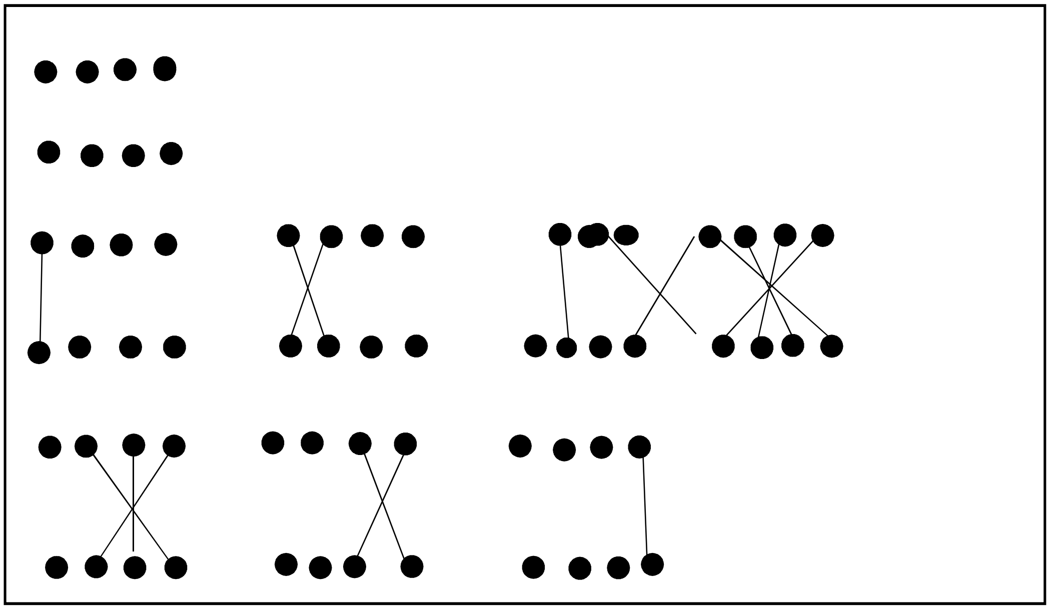


Figure-1.5

Multiplication:

323 × 245

As explained above in the diagram,

$$\begin{array}{r} 3\quad 2\quad 3 \\ \underline{\times\quad 2\quad 4\quad 5\qquad\qquad .\qquad .} \end{array}$$

$3 \times 2 \mid 3 \times 4 + 2 \times 2 \mid 3 \times 5 + 2 \times 4 + 2 \times 3 \mid 2 \times 5 + 4 \times 3 \mid 3 \times 5$

$6 \mid {}_{1}6 \mid {}_{2}9 \mid {}_{2}2 \mid {}_{1}5$

7 9 1 3 5

Thus, 323 × 245 = 79135

Division:

In special cases by simple argumentation a method of division is based on converse of Vertically & Crosswise Sūtra for multiplication.

Here, answer is 768 and multiplier 24 is given multiplicand can be found by using it.

768 ÷ 24

... …

×2 4

7 | 6 | 8

In answer 768 the number in the unit place is 8.

By using Sūtra at unit place ... × 4 = 8.

Therefore, the number in the unit place of multiplicand must be 2.

In answer 768 the number in the tens place is 6.

By using the Sūtra Vertically & Crosswise at unit place ... × 4 + 2 × 2 = 6.

Here, the sum of both the term 6 is not possible.

So by considering the sum is 16 the number in the tens place of multiplicand must be 3 and 1 must be carrying over.

So, in answer 768 the number in the hundred place must be considered as 7 - 1 = 6 instead of 7.

By using Vertically & Crosswise Sūtra at hundred place ... × 2 = 6.

Therefore, the number in the tens place of multiplicand must be 3.

Thus, the multiplicand must be 32.

i.e. 768 ÷ 24 = 32

Determinant and Matrices:

Find the value of the following determinant A of 3rd order**.**

$$|\mathrm{A}| = \det(\mathrm{A}) = \begin{array}{|ccc|} 3 & 2 & 1 \\ \hline 5 & 1 & 0 \\ 3 & 1 & 4 \end{array}$$

$$\begin{array}{ccc} 2 & 20 & 4 \end{array}$$

By using Vertically & Crosswise Sūtra for 1st and 4th row

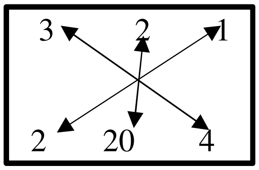


Figure -1.6

$\therefore$ Det (A) = 3(4) – 2 (20) + 1(2) = -26

The value of Fourth order determinant is also obtained by using Vertically & Crosswise Sūtra.

For finding differentiation:

It is also used for finding first order derivative and also the Successive Differentiation. I.e. second and third order derivative also can be obtained by using this Sūtra.

[1] Find derivative of $y = (x^2 - 2)(x - 7)$

As per current method, by using product rule of differentiation,

$u = x^2 - 2$, $v = x - 7$

Let u & v both be functions given in terms of x, and if y = (u) (v) then

$$\frac{dy}{dx} = u.\frac{d}{dx}(v) + v.\frac{d}{dx}(u)$$

$$\frac{d}{dx}(y) = \frac{d}{dx}[(x^2 - 2)(x - 7)] = (x^2 - 2).\frac{d}{dx}(x - 7) + (x - 7)\frac{d}{dx}(x^2 - 2)$$

$$= (x^2 - 2)(1) + (x - 7)(2x)$$

$$= (x^2 - 2) + 2x^2 - 14x$$

$$= 3x^2 - 14x - 2$$

By using Vedic Crosswise Sūtra,

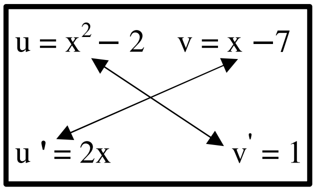


Figure: 1.7

From the above figure,

$$\therefore \frac{dy}{dx} = (x^2 - 2)(1) + (x - 7)(2x) = 3x^2 - 14x - 2$$

[2] Find $\frac{dy}{dx}$ if $y = e^x . x^e$

By using Vedic Sūtra Crosswise,

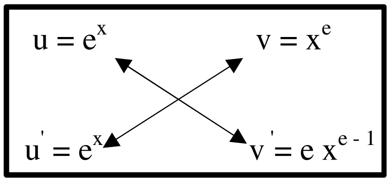


Figure-1.8(a)

From the above figure,

$$\therefore \frac{dy}{dx} = e^x . e . x^{e-1} + x^e . e^x = e^{x+1} e^x + x^e e^x$$

Taking derivative again,

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[e^{x+1}e^x + x^e e^x] = \frac{d}{dx}[e^{x+1}e^x] + \frac{d}{dx}[x^e e^x]$$

By using Crosswise Sūtra for both the terms and then add it.

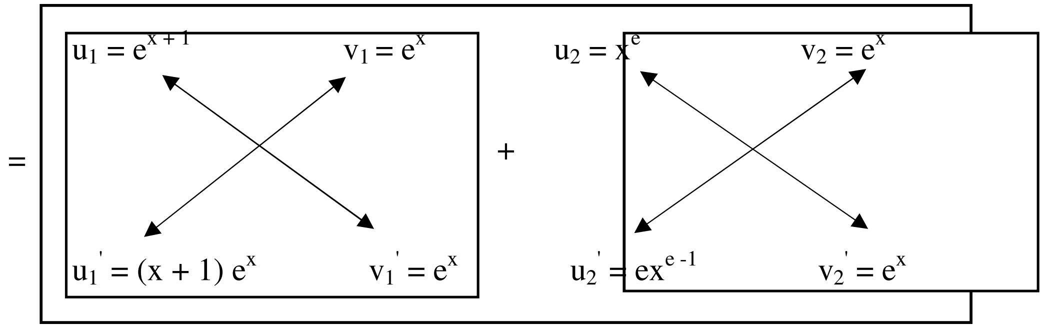


Figure-1.8(b)

From the above figure,

$$\frac{d}{dx}[e^{x+1}e^x] = [e^{x+1}e^x + (x + 1)e^x . e^x];$$

$$\frac{d}{dx}[x^e e^x] = [x^e e^x + e\,x^{e-1} e^x]$$

$$\therefore \frac{d^2y}{dx^2} = e^{x+1}e^x + (x+1)e^x.e^x + x^e e^x + e\,x^{e-1}e^x$$

Conventional Division method of finding Derivative:

Let u and v both be functions of x and if $y = \frac{u}{v}, v \neq 0$, then

$$\boxed{\frac{dy}{dx} = d\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2} \ldots\ldots.[I]}$$

E.g.:

Find $\frac{dy}{dx}$ if $y = \frac{2^x}{x^2}$

By using conventional method,

Let $u = 2^x$ and $v = x^2$

By using the above [I]

$$\therefore \frac{dy}{dx} = \frac{x^2\frac{d}{dx}(2^x) - 2^x.\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2(2^x\log 2) - 2^x.(2x)}{(x)^4}$$

$$= x^{-2}(2^x\log 2) - 2^x.(2x^{-3})$$

By Vedic Method,

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{2^x}{x^2}\right) = \frac{d}{dx}(2^x.x^{-2})$$

By using Crosswise Sūtra,

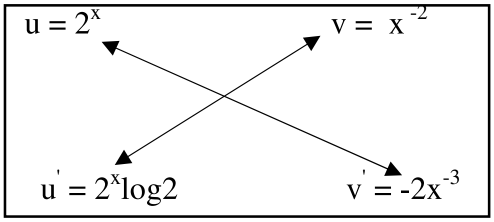


Figure: 1.9

From the above figure,

$$\therefore \frac{d}{dx}(2^x.x^{-2}) = 2^x(-2x^{-3}) + (x^{-2})(2^x\log 2)$$

$$= (x^{-2})(2^x\log 2) - 2^x(2x^{-3})$$

For integration by parts: (By using Vertically & Crosswise Sūtra)

By the knowledge of standard formula of derivative and integration; the integration of the product of two functions can be obtained.

Let $I = \int (x+1)^2 e^x dx$

Consider u = $(x + 1)^2$. Calculate its successive differentiation till we get solution zero.

Also, for another remaining function v calculate double, triple and so on.

Write all the obtained result in the figure below nd apply crosswise sutra along with alternate sign,

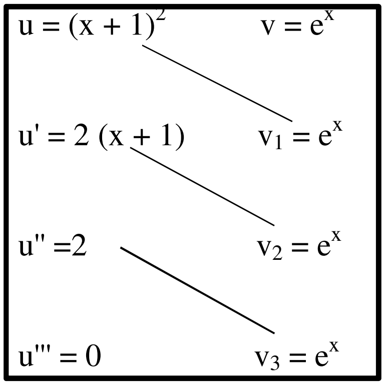


Figure: 1.10

$I = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots\ldots\ldots\ldots..$

$\therefore$ I = $[(x + 1)^2 e^x - 2(x + 1) e^x - 2e^x]$

= $e^x [(x + 1) 2 - 2(x + 1) + 2]$

Dhwandva Yoga (Duplex property of Vertically & Crosswise Sūtra:

Duplex of a number is formed in two sense (i) by squaring and (ii) by cross-multiplying. If a digit is single central digit, the duplex is a square. When the digits are even in nature, equidistance from the two sides then duplex is twice the value of diagonal multiplication. Squaring of n-digit nos., consists of either 2n or 2n-1 number of digits. Duplex of a number can be explained by considering p, q, r, s and t as digit.

$D(p) = p^2$; $D(p\,q) = 2\,p\,q$; $D(p\,q\,r) = q^2 + 2\,p\,r$

$D(p\,q\,r\,s) = 2\,p\,s + 2\,q\,r$ & $D(p\,q\,r\,s\,t) = 2\,p\,t + 2\,q\,s + r^2$

E.g.:

D (4) =16; D (1) = 1

D (3 4) = (2) (3) (4) = 24; D (1 2) = (2) (1) (2) = 4

D (123) = 2^2 +2 (1) (3) = 4 + 6 = 10; D (234) = (2) (2) (4) + $(3)^2$ = 16 + 9 = 25

And D (1234) = (2) (1) (4) + (2) (2) (3) = 20

For finding square by using duplex:

(i) Square of one digit number p. then it is p^2 = D (p)

(ii) Square of two digit number

i.e. $(p\ q)^2$ = D (p) | D (p q) | D (q)

(iii) Square of three digit number

i.e. $(p\ q\ r)^2$ = D (p) | D (p q) | D (p q r) | D (q r)| D (r)

(i v)Square of four digit number

i.e. $(p\ q\ r\ s)^2$ = D (p) |D (p q)| D (p q r) | D (p q r s) | D (q r s) | D (r s) |D (s)

E.g.

For finding square of (1 2 3 4) by using duplex property:

First find all the duplex starting from left single digit, increasing the number of digit till the digits of original numbers are used.

Then, find duplex of decreasing the number of digits till the right single digit.

By using the duplex already founded above,

$(1\ 2\ 3\ 4)^2$ = D (1) | D (1 2) | D (1 2 3) | D (1 2 3 4) |D (2 3 4) | D (3 4) |D (4)

= 1| 4 | 10 | 20 | 25 | 24 | 16

= 1|5|2|2|7|5|6

= 1522756

Thus, $(1\ 2\ 3\ 4)^2$ = 1522756

Thus, by using Duplex property of vertically & Crosswise Sūtra even square of higher digit numbers can be easily obtained.This sutra is also applicable to binary number also.

Dhwaja□ka Sūtra (Direct flag division by Vertically & Crosswise):

Meaning: Direct Flag Division.

This short cut method is applicable for division of larger decimal as well as binary numbers.

Procedure for finding Quotient part:

Split the divisor in two parts. One part which is the first number is called as new divisor and another part is the remaining numbers which is known as flag.

I.e. for two digit numbers flag is one numeral and for three digit numbers flag is two numeral. Flag digits are written above the new divisor.

Split the dividend in two parts quotient and remainder such that the number of digits in the second part i.e. remainder part and of flag both are equal.

In this method three operations are involved: multiplication, subtraction and division.

Start with dividing the first number (or sometimes with first two) of the dividend by the new divisor to get the half part of the final quotient with corresponding remainder. Write the obtained remainder in the prefix to the second digit of the dividend which forms **Gross Dividend (GD).**

Find the product of flag and the obtained new quotient which form **flag Dividend (FD).**

If flag contains more than 1 figure then by using **vertically & crosswise Sūtra** for the multiplication of flag with corresponding new quotient.

To obtain **Net Dividend (ND)** subtract FD from GD (i.e. ND = GD – FD).

If we get negative answer then reduce the quotient (which was used to find FD) by 1 and increase the remainder by the first multiplier of new multiplier and apply find new FD.

Divide ND further by the new divisor to get the next part of the final quotient with corresponding remainder.

Repeat the process until all L.H.S. numbers are used.

For finding remainder part:

In this method two operations are involved.

For one figure flag Remainder R = GD – FD.

For two figure flag FD is obtained as follows:

FD = [10 (latest two quotient × 2 - flag)
+ (latest one quotient × 1 - flag)]

And then apply R = GD – FD.

For 3 figure flag FD is obtained as follows:

FD = [100(Latest three quotient × 3 - flag)

+ 10 (Latest two quotient × 2 - flag) + (Latest one quotient × 1-flag)]

And then apply R = GD – FD.

Two digit flag:

14523 ÷123

Here, divisor is of three digits and dividend is of five digits.

∴ Two digit flag is 23.

	1 $_0$4 $_0$5	$_0$2 3
2 3	$_1$5	$_1$2
1		$_2$2
	1 ~~2~~ ~~10~~	
	1 ~~9~~	
	8	

Table: 1.1

Quotient Q = 118 and Remainder = 9

Explanation:

First digit of quotient is 1 and new divisor is 1.Therefore 1/1 then Q = 1 and R = 0.

Gross dividend (GD) = 4. And Flag Dividend (FD) = (1 × 2) = 2.

GD – FD = Net Dividend (ND)

i.e. 4 – (1 × 2) = 4 -2 = 2 (ND); 2/1 then Q = 2 and R = 0.

5 – (2 × 2 + 1 × 3) = - 2. Here, subtraction is negative.

So, decrease the Q by one and increase R by 1 × 1 = 1.

Now, 15 – (1 × 2 + 1 × 3) = 15 - 5 = 10; 10 /1 then Q = 10 and R = 0.

23 – [10{10 × 2 + 1 × 3] + (10 × 3)] = 23 - 260 = - 237.

Therefore, Q = 10 decrease by 1 and remainder increase by 1.

123 - [10{9 × 2 + 1 × 3} + 9 × 3] = 123 – [210 + 27] = 123 - 237 = - 114.

Again decrease the quotient digit by 1 and increase the remainder by 1.

Now Q = 8 and R = 2.

$223 – [10\{8 \times 2 + 1 \times 3\} + (8 \times 3)] = 223 – [190 + 24] = 223 - 214 = 9.$

Here, we get positive answer so stop. Now, consider ND = 223

Final Remainder $R = 223 – [10 \{2(8) + 3(1)\} + (3 \times 8)]$

$= 223 – [10\{19\} + 24] = 223 – [190 + 24] = 223 – 214 = 9$

Three digit flag:

$1458560 \div 1234$

Here, divisor is of 4 digits and dividend is of 7 digits.

Therefore, three flag digits are 234.

				1	$_04$	$_{\cancel{0}}3$	$_08$	$_05$	6	0
	2	3	4			$_{\cancel{1}}3$	$_18$	$_15$		
1							$_28$	$_25$		
				1	~~2~~	~~8~~	~~9~~	$_35$		
					1	~~7~~	~~8~~	$_45$		
						6	~~7~~			
							~~6~~			
							5			

Table: 1.2

$\text{Remainder} = 3560 – [100 (6 \times 2 + 6 \times 3 + 1 \times 4) + 10 (6 \times 3 + 4 \times 6) + (6 \times 4)]$

$= 3560 – [3400 + 420 + 24] = 3560 – [3844] = - 284$

Here, remainder is negative.

So again decrease the quotient digit 6 by 1 and increase the remainder by 1.

$\text{Remainder} = 4560 – [100 (5 \times 2 + 6 \times 3 + 1\times 4) +10 (5 \times 3 + 4 \times 6) + (5 \times 4)]$

$= 4560 – [3200 +390 + 20] = 4560 – [3610] = 950$

Therefore, Quotient Q = 1165 and R = 950.

Parāvartya Yojayet Sūtra:

It means first transforming it & then apply it.

The rule involving transposition leads continuous reversal of sign with the reversal of both ends i.e. Addition converts in subtraction & inversely; and also multiplication converts in division & inversely. Further it can be prolonged to the rearrangement of expressions from right side to left side & inversely and also from denominator to numerator & inversely in the given problem.

For division:

Division of the type whose dividend is greater than 1 digit & divisors are little more than power of 10. In this method considering from left to right, by leaving the first digit of the divisor and for other digits by changing the sign place them below the divisor. When the divisor consists of small digits this method is very useful.

This is also applicable to develop the process of Horner for synthetic division.

Divide 1697 by 13

$$\begin{array}{r|rrrcr} 13 & 1 & 6 & 9 & / & 7 \\ \underline{-3} & & -3 & -9 & & 0 \\ \hline & 1 & 3 & 0 & / & 7 \end{array}$$

$\therefore$ Q = 130 and R = 7

For binary division:

In computer arithmetic, by using Parāvartya Yojayet Sūtra binary division can be done same way as division of decimal number by keeping the 1st bit of divisor as it is and changing sign of the remaining bits.

For polynomial division:

$x^3 - 7x^2 + 41x - 35 \div (x - 1)$

$$\begin{array}{r|rrrr} \underline{x - 1} & 1 & -7 & 41 & -35 \\ \underline{1} & & 1 & -6 & 35 \\ \hline & 1 & -6 & 35 & 0 \end{array}$$

$\therefore$ Q = $x^2 - 6x + 35$ and R = 0

For solving simple equations:

$8x + 3 = 5x + 6$;

The problem is of the type $ax + b = cx + d$, by taking Parāvartya (8 - 5) and (6 - 3)

$$x = \frac{(6-3)}{(8-5)} = \frac{3}{3} = 1$$

Solving partial fraction:

Let

$$E = \frac{2x^2 - x - 5}{(x-3)(x-4)(x-5)} = \frac{A}{x-3} + \frac{B}{x-4} + \frac{C}{x-5}$$

Solving the above problem by using current method for finding value of A, B and C is the long procedure of solving three simultaneous equations.

But by solving with the help of Parāvartya Sūtra we can find value of A, B, C. Parāvartya Values are x = 5, 4 and 3 of C, B & A respectively.

By substituting the above values in the given expression E by neglecting the factor which is in the denominator of C, B, A respectively,

We get three values as below:

$$A = \frac{2x^2 - x - 5}{(x-4)(x-5)} = \frac{2(3)^2 - 3 - 5}{(3-4)(3-5)} = \frac{10}{2} = 5$$

$$B = \frac{2x^2 - x - 5}{(x-3)(x-5)} = \frac{2(4)^2 - 4 - 5}{(4-3)(4-5)} = \frac{23}{-1} = -23$$

$$C = \frac{2x^2 - x - 5}{(x-3)(x-4)} = \frac{2(5)^2 - 5 - 5}{(5-3)(5-4)} = \frac{40}{2} = 20$$

$$\therefore E = \frac{3x^2 - x - 4}{(x-1)(x-2)(x-3)} = \frac{5}{x-1} - \frac{23}{x-2} + \frac{20}{x-3}$$

Solving special type of Quadratic Equation:

For an equation of 2nd degree in the format as shown below:

$$\frac{m}{ex+a} + \frac{n}{fx+b} = \frac{p}{gx+c}$$

Where m, e, a, n, f, b, p, g and c can take only fixed values. Also if we have

$$\frac{m}{a} + \frac{n}{b} = \frac{p}{c};$$

We can say that x = 0 must be a solution of the above equation.

If the sum of numerators on both the sides is equal, Parāvartya is applied to combine both sides and if both are not equal then make the co-efficient of x in the denominators of both sides equal then by taking lcm & also if the condition

$$\frac{m}{e}+\frac{n}{f}=\frac{p}{g}$$

is satisfied then by using Parāvartya Yojayet Sūtra combine both sides to obtain

$$\frac{m(a-c)}{ex+a}+\frac{n(b-c)}{fx+b}=0$$

After completing the merging if the numerators of both the terms are equal then by using Śūnyam□ Sāmyasamuccaye Sūtra, $D_1 + D_2$ equals zero gives another solution for above equation.

Śūnyam□ Sāmyasamuccaye Sūtra:

Meaning: If we get same Samuccaya, which becomes 0.

Samuccaya term has a number of meaning under different context as follows:

Samuucaya in Simple equations:

In simple equation first find the factor which is contained in all expressions.

Samuccaya is that factor of the given equation.

Let $az + bz = cz + dz$ be given.

Then Samuccaya is z and solution is $z = 0$.

In expression like $(z + c)(z + d) = (z + a)(z + b)$ the product of independent term on both the sides are equal and Samuccaya is that product.

It is applicable to the equation of the type

$$\frac{m}{az+b}+\frac{m}{cz+d}=0;\ m\neq 0.$$

Where numerator of both the fractions must be same number and Samuccaya is the addition of denominators of both the fractions.

In this case Samuccaya is $az + b + cz + d = 0$

$$\text{i.e.}\,(a+c)\,z+(b+d)=0\ \therefore\ z=-\frac{b\ +d}{a\ +c}$$

Add all terms of the numerators and also add all terms of denominators. If the answer of both additions are equal that is Samuccaya and solution is that sum = 0.

If the sum of numerator i.e. ($N_1 + N_2$) and sum of the denominator multiplied by constant term i.e. K ($D_1 + D_2$) both are equal; where K = constant then by neglecting that K, Samuccaya is sum of numerator or denominator.

Examples:

$$\frac{3z+2}{5z+2} = \frac{z+1}{3z+4}$$

Sum of numerator = 4z + 3 or sum of denominator = 2 (4z + 3)

Neglecting constant term 2, both the sums are equal.

Therefore, Samuccaya is 4z + 3 and solution is z = -3/4

$$\frac{1}{3x-2} + \frac{1}{5x+1} = \frac{1}{5x-2} + \frac{1}{3x+1}$$

Here, numerator is 1 in all the fractions.

Also, sum of the denominators on both the sides = 8x -1

Therefore, Samuccaya is 8x - 1 and solution is x = 1/8

Samuccaya in Quadratic Equations:

To solve the equation of the type

$$\frac{ax+b}{cx+d} = \frac{px+q}{rx+s}$$

Where,(p - r) = (a - c); (b – d) = (s - q) .

The sum of numerator = sum of denominator = 0 is one root of the given quadratic.

If difference between numerator & denominator on both sides are equal and that difference = 0 is another root of the given equation.

E.g.:

$$\frac{2x + 5}{3x + 2} = \frac{5x + 2}{4x + 5}$$

Here, N1 + N2 = 2x + 5 + 5x + 2 = 7x + 7 = 7 (x + 1)

And D1 + D2 = 3x + 2 + 4x = 5 = 7x + 7 = 7 (x + 1)

Removing the constant term, Samuccaya is x + 1 = 0

I.e. it is as a sum of numerator (or denominator), (-1) is one root of the given.

Also, $N_1 - D_1 = x - 3$ & $N_2 - D_2 = x - 3$

Here, Samuccaya is x - 3

I.e. It is as a difference of numerator and denominator and solution is x - 3 = 0.

Therefore, 3 is another root.

Samuccaya Sūtra for special Cube:

It is applicable to the type of problems where the sum of the factors under the cube on L.H.S. is equal to Right side factor under the cube then Samuccaya is that sum and by using Samuccaya Sūtra that sum = 0 which is the solution of the given equation.

$(x - 2)^3 + (x – 4)^3 = 2 (x – 3)^3$

Solving the above equation by the current method it will take so much time.

According to Sūtra Samuccaya in 2(x - 3) by neglecting constant term,

Samuccaya is (x - 3) and solution is x - 3 = 0

I.e. x = 3 is the solution of the given problem.

This Sūtra is also applied for the special type of equations which look like biquadratic.

Ānurūpye Śūnyamanyat Sūtra:

Meaning: If out of two, one variable is in the ratio, then other variable is 0.

Solving Simultaneous equations:

If two equations given in which the ratio of the coefficient of one variable is equal to the ratio of the independent term of the two equations, then the value of other variable is zero.

Solve the following:

$345x + 240y = 1725$

$1380x + 440y = 6900$

At the first sight above example is very complicated.

The ratio of the coefficient of x is 345:1380 = 1:4 and

The ratio of the constant terms is 1725: 6900 = 1:4.

Therefore, y = 0 and we get, 345x = 1725 or 1380x = 6900

Which gives x = 5.

In Quadratic equation:

The quadratic equation of the type

$$\frac{m}{ex + a} + \frac{n}{fx + b} = \frac{p}{gx + c} + \frac{q}{hx + d}$$

where, in each terms of denominator co-efficient of x is equal to 1 and the letter other than variable x are all constant term

If value of x is not equal to 1 but if the conditions

$$\frac{m}{e} + \frac{n}{f} = \frac{p}{g} + \frac{q}{h} \ \& \ \frac{m}{a} + \frac{n}{b} = \frac{p}{c} + \frac{q}{d}$$

are satisfied then using Śūnyamanyat Sūtra one solution of the above equation is 0.

$$\frac{3}{x + 3} + \frac{5}{x + 5} = \frac{6}{x + 6} + \frac{2}{x + 2}$$

Here, both the conditions below are satisfied.

$$\frac{3}{1} + \frac{5}{1} = \frac{6}{1} + \frac{2}{1} \ \& \ \frac{3}{3} + \frac{5}{5} = \frac{6}{6} + \frac{2}{2}$$

∴ By using, Śūnyamanyat Sūtra one root of the above equation is x = 0.

Saṅkalana-Vyavakalanābhyām Sūtra:

Meaning: Addition and Subtraction

Solving Special type of Simultaneous equations:

It is applicable to the type of equations where the coefficient of x and co-efficient of y are found interchanged.

Solve the following:

$456x + 360y = 1896$

$360x + 456y = 2184$

By using Sa□kalana –Vyavakalanābhyām Sūtra,

$816x + 816y = 4080 \therefore 816(x + y) = 4080 \therefore x + y = 5$…. (i)

$96x - 96y = -288 \quad \therefore 96(x-y) = -288 \therefore x - y = -3$……… (ii)

Again Solving (i) & (ii) using by Sūtra, $2x = 2$ & $2y = 8$

$\therefore$ The solution is $x = 1$ and $y = 4$

For testing divisibility of 11:

By using Sa□kalana-Vyavakalanābhyām Sūtra testing of divisibility can be done.

First add all the digits in odd position as well as even.

Then from bigger number subtract the smaller number.

If subtraction is zero or multiple of zero then the number is divisible by 11.

Pūra□apūra□ābhyām Sūtra:

Meaning: By completion or non-completion

This Sūtra is used as completion and non-completion of quadratic, cubic and forth power etc.Using two Sūtras Parāvartya Yojayet & Pūra□apūra□ābhyām we can solve cubic equation easily by converting it in quadratic equation

Calana-Kalanābhyām Sūtra:

Meaning: Differential Calculus. i.e. 'Sequential motion'.

Tīrthajī Māhārāja called this Sūtra as calculus formula.

According to calculus formula, the first differential of the standard quadratic equation $ax^2 + bx + c = 0 = \pm\sqrt{\text{Discriminant}}$,

Where, discriminant is defined by subtracting the multiplication of twice the coefficient of second power of x and twice the independent term from the square of middle term coefficient.

$$\text{i.e.}\Delta = b^2 - (2a)(2c) \therefore \Delta = b^2 - 4ac$$

For finding roots of quadratic equation**:**

By using the above formula for Δ & applying calculus formula the given equation is break down into two linear equations.

$5x^2 – 13x – 7 = 0$.

By comparing with $ax^2 + b x + c = 0$; $a = 5$, $b = - 13$and $c = -7$.

$b^2 - 4ac = 169 – 4(5) (-7) = 169 +140 = 309$

Now by calculus formula, $10x –13 = \pm \sqrt{309}$

I.e. $10x - 13 = + \sqrt{309}$ or $10x -13 = - \sqrt{309}$

By solving this two linear equations two roots are obtained.

Factorization and Calculus:

There is a relationship between factorization of 3rd, 4th and 5th power expressions and 'Differential Calculus'.

Vya□□isama□□i□ Sūtra:

Meaning: Whole as one and one as whole

Special type of biquadratic equation in which Right side is given and Left side consists of the sum of the fourth power of two binomials whose exact middle binomial is possible which can be solved by Vya□□ihsama□□i□ Sūtra.

By using this Sūtra we can remove the odd power term of the given biquadratic equation by taking average of the two given term or by dividing its middle term in two quadratics.

Śe□a□akena Carame□a Sūtra:

Meaning: remainder with help of the end number.

Finding decimal value of 1/7

Sopāntyadvayamantyam Sūtra:

Meaning: The ultimate and twice the penultimate

The digit at the end place is ultimate & the digit previous to the ending is Penultimate.

This Sūtra is used for checking the divisibility of 4 for the given number. It is also used in simple equation and in multiplication by 12 in sandwich.

For multiplication by 12:

Put a zero before and after the given multiplicand.

First add the ultimate 0 with two time’s second last digit as penultimate.

If the sum is of two digit then carry the tens digit to the next step.

Cross out the last digit and consider the second last digit as ultimate and third last as penultimate continue the process till all digits are used.

937×12

By using the ultimate and twice the penultimate.

First write 937 as 09370

Consider last digit 0 as ultimate and 7 as penultimate.

$0 + 2(7) = 14$ carry over 1 to the next step.

Now consider 7 as ultimate and 3 as penultimate.

i.e.$7 + 2(3) = 7 + 6 = 13$ and $13 + 1 = 14$.

Carry over 1 to the next step. Consider 3 as ultimate and 9 as Penultimate.

i.e.$3 + 2(9) = 3 + 18 = 21$and $21 + 1 = 22$.

Carry over 2 to the next step.

Consider 9 as ultimate and 0 as penultimate i.e. $9 + 2(0) = 9$ and $9 + 2 = 11$.

Thus, $937\times 12 = 11244$

For divisibility by 4:

Add the last digit and twice the second last digit. If that sum is divisible by 4 then entire given no. is a multiple of four.

For checking the divisibility of 4 for 367981.

Here, $1 + 2(8) = 17$ which is not divisible by 4.

$\therefore$ 367981 is not divisible by 4.

Ekanyūnena Pūrve□a Sūtra:

It means subtract 1 from the previous one.

This Sūtra is a corollary of Nikhila□ Sūtra and converse of Ekādhikena Pūrve□a Sūtra.

Multiplications where the multiplier consist entirely of digit nines:

Number of digits of multiplicand & multiplier are equal:

Product produced by single -digit multiplier 9, by the two digit multiplier 99. If left-hand-side is one less than the multiplicand then right side is obtained by taking the complement of the obtained left side in the first step from multiplier (9, 99, 999…etc.)

2 × 9 = 1/8 = 18

8 × 99 = 17/82 = 1782

777 × 999 = 776/223 = 776223

The multiplicand consists of smaller number of digits than multiplier:

7 × 99 = 06/93 = 693

79 × 9999999 = 0000078/9999921=789999921

The multiplier consists of smaller number of digits than multiplicand:

- 29 × 9

2:9:
:-3:9
26:1
i.e. 29 × 9 = 261

- 12319 × 99

123:19:
1:24:19
12195:81
i.e. 12319 × 99 = 1219581

Binary Multiplication:

In computer arithmetic, Binary division can be done by using multiplier based on Ekanyūnena Pūrveṇa Sutrra. Just like decimal number system this Sūtra is applicable for multiplication whose multiplier is in the form of $2^n - 1$. i.e. 1, 11, 111, 1111….

In recurring Decimals:

To convert fractions whose numerator is one into its equivalent form of decimal by using the code and Ekanyūnena Pūrveṇa Sūtra.

Kevalaih (represents 143) Saptkam (seven) Gunyat:

When numerator is 1 and denominator is 7 multiplicand should be 143.

∴ By Ekanyūnena Pūrveṇa Sūtra,

143 × 999 = 142 | 857.

$$\therefore \frac{1}{7} = 0.142857$$

Kalau (13) Ksudrasasaih (represents 077):

By Ekanyūnena Pūrveṇa,

077 × 999 = 076 | 923

$$\therefore \frac{1}{13} = 0.076923$$

Kamse (17) Ksamadaha-khalairmalaih (represents 05882353):

By Ekanyūnena Pūrveṇa,

$$05882353 \times 99999999 = 05882352 \mid 04117647$$

i.e. (sixteen recurring decimal)

$$\therefore \frac{1}{17} = 0.0588235204117647$$

Guṇitasamuccayaḥ Sūtra:

Meaning: The product Sc of the sum of coefficient of the factor.

This Sub-Sūtra is used to verify the answer of the problems on multiplication, divisions and factorizations of cubic, biquadratic expressions.

This Sūtra is used along with Guṇitasamuccayaḥ Samuccayaguṇitaḥ Sub-Sūtra.

E.g.:

(y + 2) (y + 3) (y + 5) = (1 + 2) (1 + 3) (1 + 5)

$= (3)\,(4)\,(6) = 72$

Guṇakasamuccayaḥ Sūtra:

Meaning: All the multipliers

This Sūtra correlates quadratic expression with calculus.

According to Sūtra when quadratic expression is expressed as the product of two binomials then by adding all factors the 1st differential of the given quadratic expression is obtained.

Further, it can be extended for cubic and bi-quadratic expressions.

To find first differential of this expression, by applying this Sūtra convert the given quadratic equation in two linear factors such that addition of two factors is equal to the first differential.

E.g.:

$x^2 - 5x + 4 = (x - 4)\,(x - 1)$

First Differential $= D_1 = (x - 4) + (x - 1)$

$= 2x - 5 = \sum a$

1.7 Vedic Sub-Sūtras and Comparison of Vedic Methods with Algebraic Algorithms:

According to book Vedic Mathematics by TĪRTHAJĪ MĀHĀRĀJA there are 13 corollaries.

Ānurūpyeṇa Sub-Sūtra:

It implies proportionality.

For Multiplication:

For finding multiplication of those two numbers which are close to multiples of 10 the above sub-Sūtra is used. Sometimes multiplicand and multiplier both are very far away from the actual base which is multiple of 10, we get big number as deficiency. To avoid this instead of actual base select the working base which is nearer to the given multiplier and multiplicand.

By using Ānurūpyeṇa find out the proportion between the original base and working base.

After solving the problem by using Nikhilaṃ Sūtra for multiplication in the final answer only the L.H.S. multiplied /divided by the proportion obtained earlier is changed accordingly. The R.H.S. remains unaffected.

Also the right side of the final answer must consist of the no. of digit according the no. of zero contained in the theoretical base. If the theoretical base is 10 then R.H.S. of the final answer contains one digit and base is 100 then R.H.S. contains two digits.

48 × 49.

Here, if we take working base as 100 then we get very large numbers than the given original numbers.

To remove this problem we consider 100 as theoretical number and 50 as a working base.

$$\begin{array}{rr} 48 & -2 \\ \times\, 49 & -1 \\ \hline \end{array}$$

$$(48 - 1) \text{ or } (49 - 2) / (-2)(-1)$$

$$47 / 02$$

$$\frac{47}{2} = 23\frac{1}{2}$$

∴ 48 × 49 = 2352

Binary multiplication:

By Ānurūpyeṇa binary multiplication can be done by assuming working base and comparing it with theoretical base. After finding the proportion multiply the left part of the answer by that proportion same way as in decimal number multiplication.

For Cubing**:**

By Ānurūpyeṇa with the help of the knowledge of cube of 1 to 10; cube of any number more than 10 can be easily found out.

Find the cube of 98 i.e. $(98)^3$

	$(9)^3$	$(9)^2(8)$	$9(8)^2$	$(8)^3$
i.e.	729	648	576	512
		1296	1152	
	941	1	9	2

Thus, $(98)^3 = 941192$

Steps:

First write geometric ratio row-wise in 4 figures in exact proportion of first digit.

Multiply second and third proportion by 2 and write it exactly below the corresponding numbers respectively.

Then add it. The final result is obtained.

Binary cubing:

By same method as for finding cube of decimal number Ānurūpyena Sūtra can be applied to binary number.

For Factorization:

It is also used for finding one of the two factors for the factorization of given quadratic equation.

If in the given quadratic expression co-efficient of x^2 is not a unity.

To factorize the quadratic expression first factor is obtained by using the Sub-Sūtra Ānurūpyeṇa and by using Ādyamadyenāntyamantyena and the factor already obtained, second factor is obtained.

For finding first factor by Ānurūpyeṇa:

Divide the co-efficient of middle in two portions so that,

$$\text{Ratio} = \frac{\text{First coefficient of quadratic}}{\text{First part of middle coefficient}} = \frac{\text{Second part of middle coefficient}}{\text{Last coefficient of quadratic}}$$

The resulting ratio is the first factor of given quadratic expression.

E.g.:

Factorize: $4x^2 + 8x - 5$

Here, the co-efficient of middle term 8 is divided in two parts 10 and -2 such that

4:10 = -2:-5 i.e. the resulting ratio is 2:5.

Therefore, first factor of the given quadratic expression is $2x + 5$ by using Ānurūpyeṇa.

For division:

This Sub-Sūtra shows how to divide number when Nikhilaṁ Sūtra and Parāvartya Sūtras are not applicable.

Using Ānurūpyeṇa multiply or divide divisor by a factor so that either Nikhilaṁ or Parāvartya Sūtra can be applied.

After getting result multiply /divide quotient with the same factor.

E.g.

1014 ÷ 23

In the above problem the dividend is 1014 and the divisor is 23.

I. e. The divisor is not near the base of 10, 100…..

Therefore, Nikhila□ Sūtra as well as Parāvartya Sūtra both are not applicable.

So by using Ānurūpye□a if we multiply the divisor by 4 then it becomes 23 × 4 = 92 which is nearer to base 100.

So that now the problem can be solved by Nikhila□ Sūtra as explained below:

```
             |1   0   / 1   4
  23x4 = 92  |        / 8   0
             -----------------
         08  |1   0   / 9   4
             |  x  4
             | 4  0   9   2
             | 4  4   0   2
```

Ans.: Quotient Q = 44 & R= 2

After getting the result, multiply the obtained quotient by 4 which is the final quotient & if the obtained remainder is more than original divisor then divide it by that to get the final remainder.

Śi□yate Śe□asm□jñah Sub-Sūtra:

Meaning: The remainder remains constant.

The ratio in which the divisor increases, the quotient decreases proportionately, but the remainder remains constant.

For multiplication:

For multiplication of any numbers near to the base which is more than multiple of 10.

Steps:

First subtract the multiplicand and multiplier from the closest multiple 10 from the given number i.e. first find surplus considering the nearer multiple of 10 as base. Then multiply both the surplus and obtained result write in the left side of the final answer.

Add the surplus of multiplicand to multiplier and vice-versa. The obtained result is the right side of the final answer.

If the base is 200, then the right side of the answer is multiplied by 2 and if the base is 50, then divide by 2.

Examples:

I. 104 ×101

Here, Surplus of 104 and 101 is 4 and 1 respectively**.**

Since base is 100, 4 × 1 = 04 which is the left side of the final answer.

Also, 104 + 1 = 101 + 4 =105, which is the right side of the final answer.

Thus, 104 × 101 = 105/04 = 10504

II. 205 × 209

Here, surplus of 205 and 209 is 5 and 9 respectively.

The left side of the final answer is 9 × 5 = 45.

Also, 205 + 9 = 209 + 5 = 214.

Since the base is 100 × 2 = 200.The right side of the final answer is 214 × 2 = 428.

Thus, 205 × 209 = 428 / 45 = 42845

III. 53 × 58

Here, surplus of 53 and 58 is 3 and 8 respectively.

The left side of final answer is 3 × 8 = 24.

Since base is 50 carry over 2 on the left side.

Also, 53 + 8 = 58 + 3 = 61.

Since base is 100/2 = 50, the right side of final result is 61 / 2 = 30.5

Thus, the final result is 30.5 / 24 = 3074

Ādyamadyenāntyamantyena Sub-Sūtra:

Meaning: First by the first and last by the last

For Factorization:

For finding factors of given quadratic expression 1st root is found using Ānurūpyeṇa Sub-Sūtra. The 2nd root is obtained by using the already obtained first factor and Ādyamadyenāntyamantyena Sub-Sūtras follows:

$$2^{\text{nd}}\ \text{root} = \frac{\text{First term of the quadratic}}{\text{First coefficient of first root}} + \frac{\text{Last term of the quadratic}}{\text{Last coefficient of first root}}$$

E.g.:

Factorize: $4x^2 + 8x - 5$

First factor we have already obtained by using Ānurūpyeṇa is $2x + 5$.

By using, Ādyamadyenāntyamantyena Sub-Sūtra,

$$\frac{4x^2}{2x} - \frac{5}{5} = 2x - 1$$

Therefore, the second factor of the given quadratic expression is $2x - 1$.

Thus, the factors of the given quadratic expression are $(2x + 5)(2x - 1)$

Kevalaiḥ Saptakaṃ Guṇyat Sub-Sūtra:

It means the multiplicand for seven is one hundred forty three.

Here, Kevala: 143, and Sapta: 7.

This Sūtra is used for finding the decimal value of 1/7.

Once we know the value of 1/7 without any multiplication we can calculate the value of 2/7, 3/7, 4/7, 5/7 and 6/7.

Since 1/7 = 0.142857 by using this Sūtra instead of remembering 0.142857, we remember Kevala (143).

Also by using Ekanyūnena Pūrveṇa, 143 × 999 = 142857.

For finding the decimal value of 2/7 no need to find out 2 × 1/7. But first find the possibility of the last digit number.

For 2/7 since 2 × 7 = 14 the last digit must be 4 and therefore 2/7 = 0.285714.

Similarly 3/7 = 0.428571, 4/7 = 0.571428, 5/7 = 0. 714285 and 6/7 = 0.857142.

Here, all the values are in cyclic order.

Vestanam Sub-Sūtra:

Meaning: Osculators

This Sub-Sūtra is general method to find whether the given larger number (dividend) is exactly divisible by the mentioned divisor or not.

It uses the concept of Ekādhikena Pūrveṇa.

In this process Ekādhikā of the dividend is used as Positive Osculator.

Steps:

By Ekadhika multiply the unit figure in the dividend & add the remaining figures to that product. The given dividend is divisible by divisor, if the result obtained is divisible by divisor (or repetition of an earlier outcome).

Continuing the same process for finding that the obtained result is divisible by divisor or not. After few steps we get reasonably lesser no. which makes easy to decide the given larger no.is divisible or not divisible by divisor.

Examples on Ekādhika of Dividend (Positive Osculator):

The dividend which are

Ending with **9** i.e. 9, 19, 29, 39… Ekadhikas are 1, 2, 3 and 4 respectively.

Ending with **3** i.e. 3, 13, 23, 33 …to get the last digit 9 multiply all by 3

So we get 9, 39, 69, 99…….. Ekadhikas are 1, 4, 7, 10…etc.

Ending with **7** i.e. 7, 17, 27, 37…to get the last digit 9 multiply by 7

So we get 49, 119, 189, 259……. Ekadhikas are 5, 12, 19, 26…etc.

E.g.:

Check that 32896 is divisible by 29 or not.

Here, the positive osculator of 29 is 3.

Steps:

$6 \times 3 + 9 = 18 + 9 = 27$

27 × 3 + 8 = 81 + 8 = 89 ; 89 - 29 = 60 and 60 - 29 = 31; 31 - 29 = 2

2 × 3 + 2 = 6 + 2 = 8

8 × 3 + 3 = 24 + 3 = 27

Since all digits of dividend are used and also 27 is not divisible by 29.

The given number 32896 is not divisible by 29.

Answer in two lines:

3	2	8	9	6
27	8	2	27	

Here, 27 is not divisible by 29.

Therefore, 32896 is not divisible by 29.

Negative Osculators:

In the case of all divisor ending in 1, without involving bigger Ekādhika multiplier an application of Parāvartya Yojayet Sūtra is called as Negative Osculator.

Steps for finding Negative Osculator:

If the dividend ending with 1, simply drop that one. The negative osculator is the number which is previous to that one.

If the dividend is not ending with 1, then to get last digit 1, multiply it by particular number (e.g. 3 by 7, 7 by 3 and 9 by 9)

Examples of Negative Osculator:

If the dividend which are

Ending with **1** i.e. 11, 21 31, 41 ...then the negative osculator Q is 1, 2, 3, 4…

Ending with **7** i.e. 7, 17, 27, 37, ……to get last digit of the product ending with 1 multiply it by 3 so we get 21, 51, 81, 111,......etc.

The negative osculator Q is 2, 5, 8, 11…

Ending with **3** i.e. 3, 13, 23, 33…to get last digit of the product ending in 1, multiply by 7 to get 21, 91, 161, 231 …The negative osculator is 2, 9, 16, 23 …etc.

Ending with **9** i.e. 9, 19, 29, 39, 49…multiply by 9 to get last digit 1

So we get 81, 171, 261…The negative osculator Q is 8, 17, 26 …etc.

Yāvadūnam Tāvadunī Kṛtya Vargañca Yojayet Sub-Sūtra:

Meaning: “Deviate as much as deviation and add square of deviation”

i.e.: “whatever is less, lessen it and add square of it”. OR “whatever is more, add it and add the square of it”.

This Sub-Sūtra is a consequence of Nikhilaṃ Sub-Sūtra.

This Sub-Sūtra can be applicable for finding square of number and also cube of any decimal number as well as binary number of any base.

Steps for finding square of a number:

First find the surplus/deficiency based on the base and Add/subtract it from the number of which we want to find out square.

The obtained result is the left side of the final answer.

Square of surplus/deficiency is R.H.S. of the final answer.

E.g.:

Base 10

$9^2 = (9 - 1) / 1^2 = 8 / 1$ $\qquad$ $13^2 = (13 + 3) / 3^2 = 16 / 9$

$6^2 = (6 - 4) / 4^2 = 2/16 = 2 / \underset{1}{6} = 36$ $\qquad$ $18^2 = (18 + 8) / 8^2 = 26 / 64 = 26 / \underset{6}{4} = 324$

Base 100

$98^2 = (98 - 2) / 2^2 = 96 / 04 = 9604$ (as RHS requires two digits)

$107^2 = (107 + 7) / 7^2 = 114 / 49 = 11449$

$114^2 = (114 + 14) / 14^2 = 128 / (14+4) / 4^2 = 128 / 18 / 16 = 128 / \underset{1}{8} / \underset{1}{6} = 12996$

Another method:

$114^2 = (114 + 14) / 14^2 = 128 / {}_{1}96 = 12996$ (base 100)

By using the above steps also the square of binary numbers can be obtained. Yāvadūnam Sub-Sūtra is also applicable to find the cube of given quantity by modifying the method for finding square of given quantity.

Steps to find the value of a number raised to third power:

First find surplus from the nearer base. Add double of the surplus to the number which we want to find out cube.

The obtained result is the left side of the final answer.

Put down the new surplus obtained from the above and multiplied it by the initial surplus. Write it as a middle part of the final answer.

Write down the cube of initial surplus as a R.H.S. of the final answer.

Ex.:

[1] Calculate cube of number 104.i.e. $(104)^3$

Here, base is 100 and surplus is 4 by adding double of the surplus

i.e. $4 \times 2 = 8$ the final result of the L.H.S. is $104 + 8 = 112$

New surplus is 12 and initial surplus is 4.

The middle part of the final result is $12 \times 4 = 48$

The cube of initial surplus is $4^3 = 64$.

$104^3 = (104 + 8) / 12 \times 4 / 4^3$

$= 112 / 48 / 64 = 1124864$

Thus, the final result is $104^3 = 11, 24, 864$

[2] Find the cube of the number 98.i.e. $(98)^3$

$98^3 = (98 - 4) / (-2) (-6) / (-2)^3$

$= 94 / 12 / (- 8) = 941192$

Thus, the final result is $98^3 = 9, 41,192$

By this Sub-Sūtra cube of any binary number can be obtained by using the above steps for decimal numbers.

Antyayordaśake'pi Sub- Sūtra:

Meaning: The total of last digit is 10 and the previous part is exactly same.

This Sub-Sūtra is applicable for the multiplication of those two quantities whose addition of unit digits is 10.

E.g.

35 and 35; 27 and 23; 48 and 42; 148 and 152.

Here, only the last digit of first number to that of 2nd no. is 10, 100, 1000…….etc. Other than last digit remaining digits of both the multiplier and the multiplicand must be the same.

First check the sum of last digit is 10 and remaining digits are same then L.H.S. of the final answer is obtained by using Ekādhikena Pūrveṇa Sub-Sūtra and Right side of the final result is obtained by multiplication of last digits.

Examples:

57×53;

Here, the sum of the end digits of multiplier and multiplicand is $7 + 3 = 10$;

Then by Antyayordaśake'pi Sub-Sūtra

$$57 \times 53 = (5 + 1) \times 5 / 7 \times 3 = 30 / 21 = 3021$$

148×152;

Here, the sum of last two digits is $48 + 52 = 100$; then by Antyyodaśakepi Sub-Sūtra and Ekādhikena Pūrveṇa Sūtra

$$148 \times 152 = (1 + 1) \times 1 / 48 \times 52 = 2 / 2496 = 22496$$

Antyayoreva Sub-Sūtra:

It means last term only.

$$\frac{ax^2 + bx + e}{cx^2 + dx + f} = \frac{ax + b}{cx + d}$$

For equations of the above type i.e. when

$$\frac{ax + b}{cx + d} = \frac{e}{f}$$

Application of Antyayoreva Sub-Sūtra gives an accurate solution and that too involving least number of intermediate steps.

E.g.:

$$\frac{4x^2 + 3x + 2}{3x^2 + 4x + 5} = \frac{4x + 3}{3x + 4}$$

$$\therefore \frac{x(4x+3)+2}{x(3x+4)+5} = \frac{4x+3}{3x+4}$$

$$\therefore \frac{4x+3}{3x+4} = \frac{2}{5}$$

(By using Antyoyoreva Sub-Sūtra)

$\therefore$ 5 (4x + 3) = 2(3x + 4) $\therefore$ 20x + 15 = 6x + 8

$\therefore$ 14x = -7 $\therefore$ x = -1/2

Lopanasthāpanābhyām Sub-Sūtra:

Meaning: by alternate elimination and retention

Lopanasthāpanābhyām is Sub-Sūtra of Vyastisamstih Sūtra. It is very difficult to factorize the 2nd degree homogeneous equation containing three variables x, y and z.

To make it easy for solving by eliminating one variable by equating it to zero and retain in the remaining two variable by using Lopanasthāpanābhyām.Then the obtained equation with two variables can be solved by Ādyamadyenāntyamantyena Sub-Sūtra.

Repeat the process by eliminating another variable.

It is also useful in factorizing longer and harder cubic and quartic equations.

To find factors of the original expression fill in gap caused by elimination of two variables in the two set of factors.

It is also applicable for 2 or more than 2 given expression to find its H.C.F.

Vilokanam Sub-Sūtra:

Meaning: By observation

Certain forms of quadratics solve by Vilokanam Sub-Sūtra effortlessly and speedily.

Solving simple equation:

The simple equation in which left side is either of the sum

I.e. $x + \frac{1}{x}$ or the difference i.e. $x - \frac{1}{x}$ of the two reciprocals.

According to Vilokanam Sub-Sūtra, we can split the R.H.S. into the same form.

Solving Simultaneous equation:

$u + v = 9$ & $u. v = 14$

By Vilokanam, $u. v = 14$

Either $u = 2$ & $v = 7$ OR $u = 7$ and $v = 2$

Partial Fractions Problems:

These category of questions employing partial fractions of the type, where if the addition of both the denominators & the numerators are equal.

$$\frac{2x + 3}{(x + 2)(x + 1)}$$

Here, $x + 2 + x + 1 = 2x + 3$

By Vilokanam Sub-Sūtra,

$$\therefore \frac{2x + 3}{(x + 2)(x + 1)} = \frac{1}{x + 2} + \frac{1}{x + 1}$$

Guṇitasamuccayaḥ Samuccayaguṇitaḥ Sub-Sūtra:

It means the summation of the coefficients in the product is equal to the multiplication of the sum of the co-efficient of the factors.

This Sub-Sūtra is used to verify the exactness of the answers which is already found problem of division, multiplication & factorization of quadratic, cubic and bi-quadratic equations.

[1] $2x^2 - 3y^2 - 4z^2 - xy - 7yz - 2zx - 5x - 10y - 11z - 7$

$$= (2x - 3y - 4z - 7)(x + y + z + 1)$$

Here, the sum of the product = 2 - 3 - 4 - 1- 7 - 2 - 5 - 10 - 11 - 7 = - 48 and

The product of the sum = (2 - 3 - 4 - 7) (1 + 1 + 1 + 1) = (-12) (4) = - 48

Thus, by Guṇitasamuccayaḥ Samuccayaguṇitaḥ Sub Sūtra addition of co-efficient in the left side is equal to the product of sum of the co-efficient of the right side term.

[2] $x^4 + 3x^3 - 21x^2 - 83x - 60 = (x + 1)(x + 3)(x + 4)(x - 5)$

Here, the sum of the left side co-efficient = 1 + 3 - 21 - 83 - 60 = -160 and

The product of the sum = (2) (4) (5) (- 4) = -160

Thus, by Gu□itasamuccaya□ Samuccayagu□ita□ Sub-Sūtra the answer of the given problem is verified.

1.8 Outline of thesis:

The thesis will tentatively contain total seven chapters.

Chapter 1: Introduction and Mathematical Prerequisites

This chapter will introduce Vedic Mathematics and will give a brief introduction about the origin and founder of Vedic Mathematics. It will also explain the basic Sūtra and Sub-Sūtra of Vedic Mathematics. It will compare the traditional approach to mathematics with Vedic mathematics. It will discuss the benefits and applications of Vedic Mathematics.

Chapter 2: Literature Review:

It will be analysis of literature where various works done by researchers in VM will be studied in-depth. The study of various research papers, research articles, theses related to Vedic mathematics and also information from the websites which be analyzed and reviewed. This will help the researcher to identify the different areas in which Vedic math has been applied and locate the areas where there is scope for further research.

Chapter 3: Solution of Quadratic, Cubic and quartic Equations:

It will explain the scope and applications of the different Sūtras and also to what extent different Vedic Mathematics Sūtras like Vilokanam Sub-Sūtra, Sūtra Parāvartya, Śūnyam□ Sāmyasamuccaye Sūtra & Ānurūpye Śūnyamanyat Sūtra and could be useful for fast calculations in solving special types of Quadratic equations.

It will also explains how to factorize the quadratic expressions:
In one variable by mental calculation answers of which can be verified by vertically and crosswise Sūtra.

In two variables by using Ānurūpyeṇa & Ādyamadyenāntyamantyena Sub-Sūtra and also verified by Guṇitasamuccayaḥ Samuccayaguṇitaḥ Sub-Sūtra.

In three variables by using Lopanasthāpanābhyām Sub-Sūtra by converting those in quadratic form and then by using Ādyamadyenāntyamantyena and Ānurūyena. Examples of cubic expression by using Parāvartya Sūtra, Pūraṇapūraṇābhyām Sūtra are also given. It will explain with examples cubic equations but after simplifying simple equation of first degree equation by using Śūnyaṃ Sāmyasamuccaye Sūtra.

It will also examine quadratic equations in four variables by using Vedic Mathematics Lopanasthāpanābhyām Sub-Sūtra by breaking the biquadratic expression down into two quadratic equations with examples. Examples of factorization of cubic and bi-quadratic equations by using Vedic Sūtras will be explained.

Chapter 4: **Determinant and Matrices:**

It will describe the difference between Determinants and Matrices, Types of Matrices and will also explain evaluation of 3rd order, 4th order determinant by using vertically and crosswise Sūtra without writing intermediate steps; by mental calculation and also by using Elimination and Retention. It will also explain how to find out Co-factor of a matrix, Adjoint, Rank & Inverse of a matrix by using vertically and crosswise Sūtra. In this chapter in solving simultaneous equations the use of vertically and Crosswise Sūtra and to determine the consistency of linear equation and also to find area of triangle and condition of three point to be collinear will be explained.

Chapter 5: **Derivative, Integration and Solving Ordinary and Partial Linear and Non-linear Differential Equations**

It will discuss the contribution of Vedic mathematics in Calculus. It will explain the relation between the factorization and the Differential calculus by using Guṇakasamuccayaḥ Sūtra, Derivative and Calana-Kalanābhyām Sūtra, for solving successive Differentiation by using Vertically and Crosswise Sūtra, Integration by Ekādhikena Pūrveṇa Sūtra, solving integration problems on partial fraction by using Parāvartya Yojayet Sūtra and also solving Integration by parts problem by using Vertically and Crosswise Sūtra, Linear differential equations by using Maclaurin series and Taylor series, Integro-differential equations and Partial differential equations and double Integration.

Chapter 6: **Applications of Vedic Sūtras in Computer Arithmetic**

It will describe the importance of applications of Vedic mathematics in Computer. The fastest growing technology Digital Signal Processing (DSP) which is ever-present in almost every engineering discipline. Speedy calculation of arithmetic operations are of great significance in DSP. Vedic Mathematics Sūtras are applied in different arithmetic operations division, Multiplication and also for squaring and cubing which are used in applications like DSP, cryptography and computation of large calculation. Binary multiplication by using Vertically and Crosswise Sūtra, Nikhilaṃ Sūtra and Ānurūpyeṇa; Binary division by using Nikhilaṃ and Parāvartya Sūtra ; Binary squaring by using Yāvadūnam Sūtra and Vertically Crosswise Sūtra along with Duplex property & Binary cubing by using Yāvadūnam Sūtra and Ānurūpyeṇais explained with examples. It will show that Vedic multipliers based on Vertically and Crosswise Sūtra, Anurupyena, Nikhilaṃ Sūtra, Ekanyūnena Pūrveṇa Sūtra which are more efficient than the conventional multipliers like Array, Booth, Wallace tree multiplier & also Karatsuba – Ofman. It will try to show whether Vedic multipliers using different types of Vedic Sūtras achieve low power consumption, less area, high speed and fast and reduced computation.

Chapter-7 Summary and Conclusion:

Thesis will conclude with a summary of each chapters which will provide overview of the thesis, Conclusion of the whole research work, Challenges faced in current research with further suggestions and Recommendations.

Chapter: 2

Review of Literature

2.1 Introduction:

The Literature review can be defined as a study of the results concluded from the past research papers or thesis which helps the researcher for deciding an exceedingly diverse area on which the research work is not done by any researcher. These consist of latest information, learning, facts, data and research in the area done by other researchers. Reviews of such work thus become a secondary mode of information and thus while providing innovative insights related to theory and methodology do not contribute to new knowledge. For any researcher the Review of literature is vital for the clarification and proper solution of the problem selected by a researcher. The Review of past studies helps the researcher to provide a background, to develop insight for working out the methodology and design of the program for the present study to carry out the investigation on scientific line. Since effective research is based upon the past knowledge, it helps the researcher to be aware of what is already known and what is still unknown and untested so that it avoids the duplication of work of what has been already done and gives direction to the researcher to find different approaches for the future work.

2.2 Review of Secondary Resources:

The present study have collected information related the researcher’s own work from Journal-Papers, Articles, Newsletters, Books, E-Books and theses related to Vedic Mathematics for writing the review of literature, some of them are given below:

Abhijeet Kumar et.al (2012)

The research paper **“Hardware Implementation of 16 × 16 bit multiplier & square using vedic mathematics”** proposed the pattern for the architecture of multiplier & square of 16 × 16 bit which described in VHDL. There are several algorithm for multiplication like Array, Carry Save Adder, Wallace Tree, Modified booth etc. As Array multiplier needs many gates**,** it is less economical. Bits are added in carry save adder method, sequentially and provide the carry input signal to other adder which is situated in the next place and that formation takes the structure of a tree. Since implementation time is determined on the bases of no. of bits of multiplier, it becomes a drawback of this method. The limitation of Wallace Tree Method is that the design is complicated & also circuit is quite unsymmetrical. Booth multiplier gives excellent results because of the many adder cells, but has the disadvantage is high energy usage. To provide the solution of above mentioned problems the authors have designed a novel 16 bit Multiplier and Square architecture by using Vedic Mathematics Ūrdhva-Tiryagbhyām Sūtra and Duplex Property. In this paper, for 4-bit multiplication by using this Sūtra is explained with line diagram. The Square algorithm is explained by using the Duplex (D) Property. The duplex D for a 1 bit is the square of that bit, the D for 2 bit is twice their product, the D for 3 bit is the sum of double the multiplication of two corner bits with the center bit's square and the D for 4 bit is obtained by adding two times the multiplication of the two corner bits with doubling multiplication of two center bits. By using 4 × 4 blocks, architecture of 8 bits decomposed Vedic multiplier and Vedic square and also by using 8 blocks, architecture of 16 bits decomposed Vedic multiplier and Vedic Square are explained diagrammatically. The results of comparison of 16 bits Vedic multiplier and Vedic square with conventional multipliers are written in tabular form and also shown in it graphically.

Vedic multiplier & Vedic square show better results when compared to other multipliers like Array, Booth and Wallace tree multipliers. The result shows that Vedic multiplier dominates over Array, Booth and Wallace tree multipliers and Vedic Square leads over Vedic multiplier, Array, Booth and Wallace tree multipliers.

The authors conclude that 16 ×16 Vedic multiplier and Vedic Square based algorithms is better among Wallace Tree & other multipliers related to the performance, timing delay and the area of target device and also successfully implemented on FPGA.

Acharya Eke Ratna (2015)

In the article **"Mathematics Hundred Years Before and Now"** the aim is preservation of manuscript by using history of Mathematics. This article tries to connect classical and modern approach to change the archetype in mathematics by focusing upon the indications of mathematical developments. In this article the writer has explained applications of derivatives in solving quadratic equation and also explained the relation between the first order derivative and factors of quadratic expression, cubic expression and bi-quadratic expression. By giving examples, the writer has also explained application of Vertical and Crosswise Sūtra for solving successive differentiation of the product of algebraic and exponential functions and also for solving problems of Integration based on Integration by parts. Writer concludes with the help of that evidences that in 100 years before the contents have not significantly changed but algorithm and the method of solving problem has altered.

Agarwal Jyoti et al. (2013)

In their research paper **"Design & Implementation of FFT processor using Vedic multiplier with high throughput"**, they have explained about the scheme and method of FFT by applying an efficient algorithm of Vedic Math which results being speedy and dependable for calculating N-point Discrete Fourier Transform. Authors have explained how to reduce algorithm of 4 bit multiplication into 4 bit multiplication by giving an example by using Ūrdhva-Tiryagbhyām. They have also discussed about the design of a particular FFT with the help of Vedic multiplier and adder. When compare to regular Fast Fourier transform, and Vedic FFT is more efficient in term of time. A combination of FFT and Vedic Mathematics will create new developments in various fields.

Aggarwal Sarwan (Nov-2013)

In his research paper **"Observations from 'Figuring' "** by Shakuntala Devi the author wishes to establish that when 'Human computers' like Shakuntala Devi do their mental calculations, their shortcuts and tips are remarkably similar to the short cuts used in Vedic Maths secret steps. In chapter on "Digits", "Figuring" deals with multiplication table and introduces the concept of "secret step" by using Vedic Maths operations with nine and Digit Sum. Also Vedic Square considers only first nine multiples, while the book "Figuring" usually consider more multiples.

The book "Figuring" observes that

1*1 = 1, 11*11 = 121, 111*111 = 12321 and …so on 111111111*111111111 = 12345678987654321,

Which is come from symmetry in Vedic Vertically and Crosswise Sūtra for multiplication. In multiplying the number by 32 the book "Figuring" multiplies first by 4 and then by 8. That is "Divide and conquer" method. In multiplying 19 * 13, the book goes through the steps, used in base multiplication. She does not mention the word Base multiplication. The book of TĪRTHĀJĪ uses the term "Deviation" for both surplus and deficiency. In example of 63 * 48, she goes through the steps, which are used in vertically and crosswise multiplication. The book also use dividing by doubling and halving which is in Teacher's Manual. This book uses square of numbers ending in 5, we use that method in VM "By one more than previous one" but Shakuntala Devi does not refer to the Sūtra. Author has also compared this book method with Tirthaji's book for the square root, cube, cube root, Percentages, Discounts, Interest, Decimals, Fractions, Calendar and Special numbers like Pi. Author has concluded that when a human computer like Shakuntala Devi uses short cuts for mental calculations, her methods are remarkably similar to Vedic Mathematics method. Her brilliance and elegance acknowledged by the author.

Agrawal Kajal et.al (2018)

In research paper **"A Review Paper on Multiplier Algorithm for VLSI Technology"** the writers have explained many types multiplication algorithm performed by Wallace tree, Modified Booth, Karatsuba, Array and Vedic multiplier etc. which display significant part in current digital signal processing along with its many applications. With advances technology, many researchers are trying to design multiplier with combination of Low power and high speed VLSI implementation. The popularity of traditional Array multiplier is due to its consistent structure which is constructed on algorithm of adding & shifting and minimum complexity. Main disadvantages are because of numerous logic gates it uses more chip area. It is restricted to 16-bits because of high power consumption. Wallace tree multiplier has developed an effective hardware by implementing a digital circuit which carries out multiplication of two bits. The main advantage is that it becomes more evident for more than 16- bits and disadvantage is that because of irregular structure of logarithmic depth reduction tree based CSA, its design and layout is difficult. Booth multiplier is best for signed numbers in which algorithm of multiplication is done by taking 2's complement of binary nos. which we want to multiply. As this multiplier is able to scan 3 bits at a time, it reduces the delay which gives greater benefit. The multiplier's disadvantage is that because of more power consumption the efficiency of the system reduces. In a fast multiplication algorithm Karatsuba by using a divide and conquers approach input is divided in two parts: Most significant Half & Least Significant Half. The benefit of Karatsuba algorithm is that it decreases number of multipliers by replacing them with adder and the drawback is that if width of input is less than 16-bit, it is optimum. Vedic multiplier is based on two Vedic mathematics Sūtras: Ūrdhva-Tiryagbhyām (Crosswise and vertically) and Nikhila□ Navataścaramam□ Dasatah Sūtra (All from 9 and last from 10) which are applied for multiplication of large number by large number. Vedic Multiplier can solve the larger number multiplication with minimum delay which is the beneficial part. The disadvantage is that the multiplication process becomes tedious as the number of bits increases. For FPGA implementation, the authors have taken 16-bit multiplication on array, Wallace, Booth, Karatsuba algorithm and compare it with Vedic algorithm with respect to delay, energy consumption**,** memory & number of LUTs. The result shows that Vedic multiplier has low delay, low power consumption, low memory requirement and consumes less number of LUTs compared to other multipliers. Thus, they have concluded that for FPGA implementation, Vedic multiplier is more efficient for 16-bit multiplication among all multipliers and can give speedier computation when it is used in the generic processors and DSP applications.

Anju, Agrawal V.K. (2013)

In research paper "**FPGA implementation of low power & high speed multiplier using Vedic mathematics"**, writer recommends the design of 8 by 8 bit multiplier by using Vedic Vertically and Crosswise Sūtra.

The recommended structural design decomposes and dissipates into 4 by 4. Ūrdhva-Tiryagbhyām Sūtra for 8 by 8 bit multiplication have been done. This Vedic Sūtra applied to binary numbers using carry save adders. In section II, just like decimal number multiplication, Ūrdhva- Tiryagbhyām Sūtra is explained by taking an example of binary multiplication of 2 by 2 digits, 3 by 3 digits and 4 by 4 with line diagram by using Ūrdhva Tiryagbhyām Sūtra. In section III, hardware design for 4 by 4 & 8 by 8 bit Vedic multiplier modules are displayed. By using Vedic Multiplier partial product additions & generations are executed simultaneously. The block diagram of 4 by 4 and 8 by 8 bit Vedic multipliers are shown in this paper. In section IV, Vedic multiplier of 8 by 8 bit is implemented in VHDL.

After simulation results and after synthesizing outcome of Vedic multiplier with logical source application, the researcher concluded that the computation of pathway delay in time for the given

8 by 8 bit Vedic multiplier is found to be smaller than traditional system. Even in respect to the area, the new multiplier is more efficient than the traditional one.

Babaji D.K. (2012)

In a Research paper **"Solving system of linear equation using Parāvartya rule"** the writer has shown the method of solving given two linear equations with the help of Parāvartya Yojayet Sūtra and how to apply it for solving n linear equations through some examples on the system of three linear equations, four linear equations and eight linear equations.

The writer deduces that the method of solving two linear equations by using Parāvartya Yojayet Sūtra is comparable with Cramer Rule and solving system of three linear equations is beneficial over Gauss elimination method and LU Decomposition method. Also for solving n linear equations by Parāvartya Yojayet Sūtra, if n is odd, Sūtra is applied (n-1)/2 times and if n is even, it is applied n/2 times. The writer is exploring on application of Parāvartya rule more time to avoid substitution in between the steps in system of three and four linear equations.

Bajaj Ronak (2005)

In the book **"Vedic Mathematics, The Problem Solver",** Chapter 1 says that when one applies Sūtra to certain questions requiring creativity and rationality, this found that there is an increase in a person's self-confidence. The one thing common among all math prodigies since 15th century is the desire to excel by developing innovative methods to approach the given problem. In Vedic methodology, solution to any given problem can be arrived at in two different ways-one being the implementation of the general approach & the other one is identifying the arrangement of variables and application of situation specific formulas.

As per his opinion, a significant and interesting invention which has led to various applications in all the disciplines is the development of VM approach. This wide applicability can be explained by the quick & insightful solutions obtained through basic minimum arithmetic procedure.

The author has tried to compare each Vedic Sūtra with algebraic proof by providing sufficient examples.

Barik RanjanKumar, Pradhan Manoranjan (2015)

This article presents **"Area-Time Efficient Square Architecture"** use of Vedic Mathematics Yāvadūnam Sūtra for designing a new square architecture. This is first explained in decimal number system. This is applied to obtain square of a number which is close to base of power of 10 by considering two cases: First find nearest base of the given number. (i) If base is bigger than the given number, find its insufficiency using number deduction w.r.t base. The left part of the result (LPR) is obtained by deducting deficit from the original number (i.e. LPR = number – deficit) and right part of the result (RPR) is square of deficit i.e. RPR = $(\text{Deficit})^2$. (ii)If the given number is greater than base, find its surplus by subtracting the base from the number. The left part of answer (LPR) is obtained by adding the number with surplus (i.e. LPR = number + surplus) and right part of the answer (RPR) is obtained by squaring the surplus (i.e. RPR = $(\text{surplus})^2$. In both the cases (i) The number of digits in right block must be equal to the number digits in power of base (ii) Concatenate LPR and RPR to find final answer of finding square of given number. The Yāvadūnam Sūtra applied in the decimal number system is explained by giving examples in different situations. In a similar way, the applicability can be extended to binary system. In this method to calculate product of any number with itself, the base is taken as 2^n i.e. the base of binary number is taken as 100 and 10000 for 2 - bit and for 4 - bit respectively. In this article authors have considered only the deficit case of Vedic Yāvadūnam Sūtra where the number is less than the base and followed steps as per specified Sūtra. They have proved that to find complement of 2's is correspondent to subtracting binary integer (n bit) from 2^n. Considering one illustration of a number with 4 - bit and then the proposed design is evaluated in terms of path delay and area for bits of higher power with base 2. Observation of both the techniques existing and recommended already obtained by the researchers are discussed in review of literature.

After the comparisons, result shows that the proposed Vedic Squaring architecture seem to have both speed and area improvements over existing architectures and so proposed architecture may be useful for high performance computer graphics, cryptography, ALU circuits, and many digital image processing applications.

Basanna, M

In the book **"Vedic Mathematics"** writer has started with explanation of complement from 10 and from 9, subtraction from 10, 100, 1000…..which are useful in Vedic Maths. In Vedic

Mathematics addition can be done from left to right also. By converting the number into Vinculum, subtraction is done. Multiplication with special cases like multiplication by 11, 111, 1111……..,multiplication by 12, Ekādhikena Pūrve□a multiplication for last digits adding to 10, squaring a number ending with 5, multiplying a number with multiplier having only 9's (9, 99,999……). Nikhila□ System-Base system is explained and Nikhila□ multiplication when both multiplicand and multiplier above the base, below the base, one above the base and other below the base. Nikhilaam Sūtra is applied only for special cases but Urdhva- Tiryagbhyam Sūtra (Vertically and crosswise) is for multiplication of any two nos. and it is also explained by line diagram. The example of multiplication 95 × 95 is explained by Ekādhikena Pūrve□a Sūtra and Ūrdhva-Tiryagbhyām Sūtra, Nikhila□ Sūtra. Various method of Division in VM are Nikhila□ and Parāvartya Division i.e. Transpose and Apply, Straight division by using the combination of Vedic Sūtra Dhvaja□ka and Ūrdhva-Tiryagbhyām Sūtra. Division with 1 flag, 2 flag and 3 flag digit is explained with three methods by Nikhila□ division, Parāvartya division and by straight division. Out of them first two are simplest and easiest methods. Square by Ekādhikena Pūrve□a and Yāvadūnam (deviate as much as deviation and add square of the deviation) method finding square of number ending in Five. If number is large Ūrdhva - Tiryagbhyām Sūtra & Duplex or Dwandvayoga can be used. The example of square of 95 is explained by Ekādhikena Pūrve□a, Yāvadūnam and by Dwandvayoga and out of three methods of finding square Ekādhikena Pūrve□a is best.

Also for finding cube Yāvadūnam, for Recurring Decimal Ekādhikena Pūrve□a helps to convert vulgar functions of the type 1/p9, where p = 1, 2, 3, .9 into recurring decimal. Equation by Parāvartya for different types.

Vedic Mathematics also gives simple and elegant method to find the solution for the category of:

Equal ratio of co-efficient of x (or y) and constants by using "if one is in the ratio, the other one is zero.

Co-efficient of x and y are interchanged by using Vedic Sūtra "By addition and subtraction."

Bengali Saurabh (2011)

In the thesis**, "Vedic Mathematics & its applications in computer arithmetic"**, according to author the foremost purpose of VM is to convert the complicated computations in a very simple form in such a way that one can solve it mentally without writing any intermediate steps. Just like in the decimal number system, VM Sūtras are also applicable to binary number system for evaluating arithmetic operations using computer. Since ALU plays a central role for computations, there is a necessity for ALU which is speedy & effective at the same time. In the 2nd part, author has enlightened various Sūtras with their meaning & its applicability. By using Nikhila□ Sūtra and the Ānurūpye□a Sūtra for solving the case of the special types of multiplication and Urdhva-Tiryagbyam Sūtra for all types of multiplication for decimal numbers as well as for polynomials, decimal squaring by Dwandwayoga means Duplex used for a multiplication of digit by itself, division by Nikhila□, Parāvartya & Ānurūpye Śūnyamanyat Sūtra is explained briefly by giving examples. In 3rd part he has explained the product of two numbers as well as product of a number with itself for binary system. Nikhil□ & Ānurūpye□aboth are not universal methods in binary system as one of the multiplier must have a base in power of 2 but Ūrdhva-Tiryagbhyām for multiplication is universal method chosen for implementation. Multiplication and Squaring in binary system using Ūrdhav-Tiryagbhyām by giving example is explained very nicely. In this section, UT multiplier has been evaluated with conventional Wallace tree in terms of Time, Area and Power. In chapter 4, binary division by using Nikhila□ & Dhvajanka Sūtra have compared with conventional division in terms of Time, Area and Power. Chapter 5 & 6 discusses various methods like Taylor's**,** approximation of polynomial, Euler's & Interpolation available in hardware for calculation of trigonometric functions sine and cosine, out of which author has selected Polynomial Approximation and Interpolation as the best method**.**

Author has highlighted the possibility of integration of elementary quantitative techniques & VM approach.

BHĀRATĪ KR□□□A TĪRTHAJĪ MĀHĀRĀJA Śankarāchārya of Govardhan Matha, Puri (2013)

In the book **"Vedic Mathematics-** by SWĀMĪ ŚRĪ BHĀRATĪ KR□□□A TĪRTHAJĪ MAHĀRĀJA, as per General Editor V.S. Agrawala, Swamiji has tried in all his aspects to explain practical implementations of Vedic Sūtras across diverse disciplines in the mathematics like Arithmetic, Calculus-Differential & Integral, as well as astronomy. List of Vedic Mathematics Sūtras and Sub-Sūtras from Vedas which were compiled from loose and weak secondary sources of the manuscript. Vedic Sūtras and their applications mainly deal with work on boring & burdensome computations along with executing them mentally having great range. This book explains how to apply VM Sūtras and Sub-Sūtras to different topics of Mathematics and connectivity with it is explained very nicely. Author has covered chapters in which multiplication for all types by Ūrdhva-Tiryagbhyām Sūtra & product of bigger numbers by Nikhila□ Sūtra & Ānurūpye Śūnyamanyat, Sūtra Parāvartya & Sūtra Nikhila□ for division; for solving simple equation by Parāvartya Sūtra & Sūtra Sunyam Samuccaya; Ānurūpye Śūnyamanyat & Ādyamādyenāntyamantyena Sub-Sūtra for the factorization of quadratic equations in 2 variables & for 3 or more than 3 variables by the process of using vedic maths Sūtras like Lopanasthāpanābhyām, Ādyamādyenānantyamantyena & Ānurūpye Śūnyamanyat; Parāvartya Yojayet, Lopanasthāpanābhyām & Pūra□apūra□ābhyām for solving cubic equation; Solution of simultaneous equations is explained with the help of Sa□kalana-vyavakalanabhyām Sūtra & Ānurūpye Śūnyamanyat. Relationship between differential calculus and factors of that polynomial is explained by taking examples. If one knows the factors of the given polynomial then one can find its 1st, 2nd, 3rd and so on differentials & vice versa by using Gunaka-Samuccaya & Ādyamadyenāntyamantyena & the problems of Integration on Partial fraction by using Parāvartya Yojayet Sūtra in details. Author has also included the topics like finding square, cube, square root, cube root, recurring decimal & divisibility test by using particular related Sūtras.

Bhatia Dhaval (2009)

In the book **"Vedic Mathematics Made Easy"** the author has explained basic level of Vedic Maths in a very simplified manner. He has divided the Vedic Mathematics Techniques in two types. The first type focuses on speedy and reliable but are only suitable for certain pair of numbers. On the other hand, applications of the second type are potentially infinite**.** To make maths interesting, he has given a special chapter on 'Mental Magic' in which he has given the techniques like how to predict a person's date of birth, how to predict a person's pocket money. The modern day requirement of aptitude tests include logical reasoning questions which are covered in a very interesting way by the author in form of illustrations. For the scholars who are preparing for exams which test their aptitude and subject knowledge as well as speed like Management entrance, Engineering entrance tests, Union Public Service exams, Graduate Aptitude Tests and others at national level and also for admissions to foreign universities like, Graduate management Aptitude Test. Vedic Mathematics techniques are useful in solving problems which are mathematical in a speedy manner and to crack these exams.

Can Eyupoglua (2015)

In his research paper, **“Investigation of the Performance of Nikhila□ Multiplication Algorithm”** the researcher has introduced multiplication which is very essential operation for binary system. Also used for other operations like computing Reciprocal, Division and Squaring. In applications of DSP like image processing, correlation, convolution and frequency analysis multiplication is very important. Considering multiplicand and multiplier with equal length multiplication by using Vedic Nikhila□ algorithm is performed. In the second section algorithm of multiplication for the Nikhila□ is elucidated. In third section, the performance of Nikhila□ Algorithm for 4, 8, 16 and 32 bits length are studied in relation to numbers related to multiplication as well as the estimated processing period of time along with their performance analysis for diverse length bits is shown graphically and concluded that with the increment of length of bit, a higher period is requied for operational processing. The total process time is inversely proportional to the processing speed.

After comparing Nikhila□ Multiplication Algorithm to the regular technique, Nikhila□ Algorithm with respect to proliferation & processing period has better result than regular multiplication.

Chandra Rohan (May 2015)

In his research article **“Extension Method of Computing Squares of Two Digit Numbers” (Square Made Easy)** shows a simpler method of computing two digit squares of the form

$10x + y$ where, $0 \le x \le 9$ & $5 \le y \le 9$, x and y being integers.

The paper presents an extremely easy and short method using Vedic Maths to demonstrate of squaring two digit numbers. Author has explained the technique by giving example the importance of being able to square a 2 digit number quickly. The shortcuts given by Vedic Maths are accepted across the board. It is the deviation method in which the distance of a particular number from the closest multiple of ten and add and subtract the distance from the number and also multiply the two numbers and add the square of the distance to get the final answer. In this article the method is explained based on the algebraic manipulation by taking

$(10x + y)^2 = 100x (x + 1) + 20 (x + 1) (y - 5) + (10 - y)^2$, where $5 < y < 9$

Here, R.H.S. = $100x^2 + 20xy + y^2$ which is equal to L.H.S.

The pattern based on the above formula is explained to find value of A, B and C and then by adding three numbers together answer of a square of given number is obtained easily.

Here, $A = 100 x (x + 1)$, $B = 20 (x + 1) (y - 5)$ and $C = (10 - y)^2$.

For finding B first take $(x + 1) \times 10$ and then double it.

By using the new pattern show how far $10 x + y$ is far from $10 x + 5$ and multiply it by that distance.

Thus, using the above formula, we reduce a multiplication problem to that of a simple addition problem.

Chaudhary Ila, Kularia Deepika (2016)

In their research paper **"Design of 64 bit High Speed Vedic Multiplier"** the writers have discussed about Ūrdhva-Tiryagbhyām for 4 × 4 bit, application of Nikhila□ Sūtra for the product of two numbers (4 bit) with the help of a diagram, various methods used for partial products like Zero padding, Ripple Carry adder, Carry skip adder, Kogge stone adder and Carry Look Ahead adder. The recommended multiplier structural design of Vedic multiplier with 64 by 64 bit & for partial products half adder is used. In this paper Partial product with previous carry is explained with figure. The VHDL code of Vedic multiplier with 64 bit was synthesized & comparison results of Vedic multiplier with 8 bit and 64 bit Array multiplier, Booth multiplier and multiplier employing vedic method with respect to speed and space is represented by tabular form.

In the end they have concluded that Vedic multiplier of 64 × 64 bit has more speed and small area as compared to array and Booth multiplier.

Chidgupkar Purushottam, Karad Mangesh (2004)

The research paper **"The implementation of vedic algorithm in digital signal processing" describes** the fastest growing technology DSP is present across any engineering discipline. Quicker addition and multiplications are greatly valuable. The authors highlights the applications of product calculation procedure grounded upon VM. Also the execution of above procedure on several data processing systems is discussed. Authors have concluded that compared to conventional mathematical methods, the VM execution on data processors results in faster &

easy computation along with significant reduction in processing time. Vedic algorithms on specially designed BCD architecture for further savings in processing time by increasing the output of data processor.

Further research project may be done to increase the inclusion of Vedic methods in academia as well as industries.

D.Rajasekar, E.Anbalagan (2016)

In their Research paper **"Design of High Speed Optimized Vedic Multiplication Techniques"**, they have proposed the design of ALU by building upon the fundamental formulas of Vedic Maths & enhancing the effectiveness of the same. Vertically and crosswise method is an effective way to find product represented by writing multiplicand and multiplier on two consecutive side of a square which comprises of columns& rows. In this paper 2 separate methods of coding are utilized to execute the recommended product operation. The writer has obtained the delay comparison for different multipliers. The outcome as produced by recommended VM was shown to be quicker as compared to Karatsuba algorithm.

Dafe Suraj et al. (2015)

In research paper **"Design of high speed multiplier using vedic mathematics techniques"** writers described how the necessity of effective processors has been at high demand these days. Since processor speed depends on multiplier, which mostly power dissipation and more delay.But Vedic multiplier based on Ūrdhva-Tiryagbhyām algorithm for multiplication results in minimum delay. Author has implemented the basic logic for 4 bit Vedic multiplier in which less hardware is used.

Just like the method of vertically and crosswise we use for decimal numbers. It can also be used along with the basic concept through which partial product can be done. For 4 × 4 Vedic multiplier we get resultant result of 8 bit. In general for n inputs of Vedic multiplier we get resultant result of 2n bits. The techniques of that Vedic multiplier is shown by figure.

Compared to other multipliers, when input rises, area & time period also rises. The 4 × 4 VM can be assembled from four 2 × 2 VM which consist two individual input which are internally connected, two 4 bit full adder and one half adder. For each 4 bit input make the group of 2 bit and that will result in Vertically & Crosswise product which is shown by the partial product in the block diagram of 4 × 4.

After comparing the result of simulation of Vedic and traditional multiplier author concluded VM to be fast compared to traditional booth multiplier. Authors have concluded that among all other multipliers Ūrdhva-Tiryagbhyām Sūtra is best suited with respect to speed and delay.

Dani S.G. (2017)

His research article, is titled **"Vedic Maths': myths and facts".** According to the author, as per propaganda report Vedic Mathematics could be considered a strong computational method with least efforts which makes it more amazing & interesting. Additionally the range of questions that one can encounter easily is very high when compared to the amount of efforts made to memorize the basic formulas initially. This VM approach eliminates all the unnecessary intermediate calculations to an extent that the question can be solved mentally. The question whether the Sūtras are from Vedas is answered by author in a tone that reflects a surprising finding that the similarity between the Vedic Sūtras & Atharvaveda is very little. One cannot find the exact Vedic methods in the texts of Vedas. Therefore, the authenticity of relationship between them cannot be fully validated. Do Vedic principles convey mathematical idea? The author says, the Ekādhikena Pūrveṇa Sūtra can be left open to multiple interpretations (E.g.it can also be about family planning). It is not necessarily restricted to a quantitative understanding. Author concludes that several principles of VM cannot be utilized to formulate a specific methodology due to their broad sense of guidelines. Therefore, this situation is of little help other than initiating a philosophical discussion among the audience. Author has said that by mere possession of tools one cannot create a masterpiece. Similarly Vedic shortcuts alone will not lead to quick & accurate results. Along with them, pattern identification & implementation of specific methodology are the required ingredients for a perfect dish. Hence, sufficient care must be taken to give due credits & not to attribute the success in this endeavor to a certain specific set of tools. The author finds the claim suspicious especially due to the introduction of certain concepts of maths much later than the applied mathematics mentioned in the Vedic literature. Also there were multiple claims of discovery during that period across Europe. According to the author the development of VM approach focuses on tricks & shortcuts leading to quick computation under some special cases only. He is not convinced about the depth of the fundamentals covered & its applicability to computational sciences. On the contrary author warns about the dangerous effect of the development of a person’s quantitative aptitude.

In the end, the author has written that although Vedic approach is demanding in efforts & lucid instructions, their applications are difficult to ignore.

Das S. (1994)

In his paper **“Solutions of Linear Algebraic Equations and Sum of Fraction-Additions using Sūtra Method”**, he explains the salient features of this unique Mathematics which explains the enigmatic meaning of five formulae and explains their procedures with example. The examples cover the solution of linear algebraic equation in one, two and three variables with rational, irrational and mixed coefficients and determination of sums of fractions–addition using the Vedic Mathematics Sūtras Method. He has used five Sūtras in his investigations. Sūtra Method is an alternative style of expressing mathematical terminology.According to him finally the success of Vedic Mathematics provides enough motivation.

Deshpande Nishant, Mahajan Rashmi (2014)

In their research paper “**Ancient Indian vedic mathematics based multiplier design for high speed & low processor”** writers describe the architecture and operation of 4×4 multiplier by using Vedic Ūrdhva-Tiryagbhyām & Nikhila□ Sūtra (All from 9 last from 10) using EDA (Electronic Design Automation). All multipliers are an important component in DSP systems. Multiplier architectures are Serial multiplier, Parallel multiplier and Serial-Parallel multiplier. The multiplier designed by using Vedic Sūtra is known as Vedic Multiplier which consumes less energy in addition to being swift.

In section –II, related work on proposed multiplier is discussed by using two techniques adding and shifting procedure.

In section III, Multiplication by using Vedic mathematics Technique Ūrdhva-Tiryagbhyām is explained by taking example of two decimal numbers. It is also applied to 2 × 2 bit multiplier. Implementation of 2 × 2 multiplier is displayed by 2 by 2 Block Diagram. For a 2 bit multiplication, result obtained is 4 bit.

Results show that recommended VM design displays speedy improvement. After simulation result in terms of time required to get result Vedic multiplier takes lower time with respect to traditional ones.

Dhanave M., Kangale M. (2014)

This research paper **"The Implementation of Vedic Mathematics to Algebra and Geometry"** has discussed the study of mathematics in two directions outer and inner. According to them, applying outer direction is practical, useful and beneficial and the inner direction reminds us the basis of entire creation like our origin, the unity and supreme. Execution of various Sūtras has been portrayed in a very lucid manner covering certain aspects of Algebra and Geometry by explaining factorization of polynomial expression of two or more variables by using Ādyamadyenāntyamantyena Sūtra which means first by first and last by last. It explain this also by taking examples to find area of given shape which can be solved either by addition or by subtraction.

The authors have concluded that Sūtras are both objective and subjective in their character.

Dixit Sajay (May 2015)

In article **"Power of numbers"** writer has expanded the concept of finding third and fourth power of two digit numbers by using binomial theorem which was explained in treatise by Tīrthajī in chapter: Elementary Squaring and Cubing. In this article the author has covered the area of finding second, third, fourth and fifth Power of two digit numbers, finding second, third and fourth power of three digit numbers and finding second and third power of four digit numbers. Pascal's triangle gives the coefficients of any power for binomials. Square, cube, fourth power and fifth power of two digit numbers by using co-efficient of power 2, 3, 4 and 5 of Pascal's triangle respectively. If both the digits are less than 5, by using Ānurūpyeṇa Śūnyamanyat Sūtra and if either one or both the digits are greater than 5, by using Vinculum complexity can be reduced.

In this paper the method to find square, cube, fourth power and fifth power of two digit number is explained by finding part of numbers which is generally one more than the power we want to find out and taking ratio of two digit number starting with first part either from right or from left writing respective power of the digit and other parts are filled up by multiplying ratio with the previous part by using Ānurūpye Śūnyamanyat Sūtra defining it as X and considering the co-efficient as Y by using Pascal's triangle for respective power. Then by multiplying X with corresponding Y and keeping only one digit in each part and extra digit carry over to the left side, we get the answer of respective power of two digit number. By using same method for finding respective power of three digit number by taking ratio in three combinations of two digit and four digit number by taking ratio in six combinations of two digit can be obtained.

Also number of parts can be found by subtracting 1 from the no. of digits, multiplying it with respective power and adding 1 to the obtained number, the result we get is the number of parts. Few examples are also explained very nicely.

Elakkiya J., Mathan. N. (2015)

In their research paper **"Survey on Performance of Vedic Multiplier"** they have discussed different types of VM Sūtras like Ūrdhva-Tiryagbhyām, Nikhilaṃ, Parāvartya Yojayet & Guṇaka-Samuccaya Sūtra. They have surveyed on the efficiency of the design methodology of Vedic multiplier compared to conventional multiplier. They have taken a review of six papers related to the Vedic multiplier, and concluded that Vedic multipliers can be implemented in different field of applications.

This paper has analyzed the Vedic multipliers by using different types of Vedic Sūtras to achieve fast operation with minimal efforts.

Fernandes Chilton, Bokar Samarth (2013)

In their research paper **"Application of Vedic Mathematics in Computer Architecture"** the authors have shown the use of Vedic Mathematics in Computers. Section–II provides the review of Vedic Sūtras and Sub-Sūtras as given by TĪRTHAJĪ MĀHĀRĀJA in 1960, and discusses the beauty of these aphorisms and their corollaries is that they simplify the cumbersome process of calculations into a simple technique. The authors have taken into account two primary Sūtras Nikhilaṃ & Ūrdhva-Tiryagbhyām Sūtra.

The working algorithm of Nikhilaṃ is given as $p = x.\ y$,

Product of two n bit numbers x and y as

$$p = 10^n \{(y - x_1)\} + \{x_1 y_1\}$$

OR

$p = 10^n \{(x - y_1)\} + \{x_1 y_1\}$, where $x_1 = 10^n - x$ and $y_1 = 10^n - y$.

Section –III is devoted to the use of high speed maths in the field of IC layout and planning. The fields such as VLSI encoding and Discrete Fourier Transform use multiplication and Division, Squares and Cubes. Conventional methods are lengthy and cumbersome. Vedic Maths provides a simple solution to it which is compared in Table 1 of section V.

Gaikwad Kaustubh, Chavan Mahesh (2015)

In the research paper **"Vedic mathematics for digital signal processing operations: A review"** the writers have gone through related investigation about the utility & recent trends of VM approach in the field of DSP by referring forty surveys. In engineering a fast growing area is Digital Signal Processing (DSP) in which quick arithmetic operations & DSP intimation have prime significance. To obtain this objective, the application of VM algorithm in finite & infinite impulse response filter implementation on FPGA, Architecture for fast polynomial division, A block convolution process, The systolic matrix multiplier, Square based high performance multiplier, A multiply and accumulate (MAC) unit, Reconfigurable design of N×N multiplier, Reconfigurable Fast Fourier Transform (FFT) design and Implementation, Implementation of Vedic algorithm on 8085 and 8086 micro- processor. After study of all these research papers, the authors have concluded that Vedic Mathematics is implemented across all the disciplines of applied science in which effective algorithm can be applied to design digital filters for addressing pressing issues that we face globally.

By using VM approach there are chances to solve problems of poor compactness, high time period, large input requirements & cumbersome construction.

Glover James (Oct 2013),

In research article **"Extending the application of Vedic Mathematics Sūtras" FIMA,** the author has compared conventional method with Vedic method. In conventional method the line equation with gradient whose one co-ordinate is known. Let (x_1, y_1) be given point, then straight line equation with its slope m is

$$y - y_1 = m (x - x_1)$$

The Vedic alternative method gives the same equation but in a slightly different form as $mx - y = mx_1 - y_1$. By using the Vedic Parāvartya Yojayet Sūtra in finding straight line equation in the form of ax - by = c. The Vedic Sūtra also applied for finding area of parallelogram. Two cases are explained in this paper. One in which a parallelogram has one vertex at the origin and the two adjacent corners have co-ordinate. By using area formula using two co-ordinates the area can be found. Second case is a parallelogram with no corner is on the origin and three vertices are given then transposing the parallelogram so that one vertex is at the origin. This is done by subtraction of one point from other two. Translation gives two points. The area can be found out by using the Vedic area formula the product of mean – product of extremes, area can be found. The conventional method requires working out the determinant of a 3rd order matrix. The advantage of Vedic Method over the conventional method is that it is very simple to remember and also reduces the number of arithmetic operations involved by a half. When applied to the unit square, the resulting shape, which, for non-zero determinants will always be a parallelogram, has an area equal to that determinant. Since every triangle can be considered as half a parallelogram, the Sūtra can be applied to problems of finding areas of triangle. This application, for finding areas of parallelograms, give the basis of a one line proof of Pythagoras' theorem.

Glover James (2013)

In research article **"The Psychology of Vedic Mathematics- Examples of universal thought patterns"** the writer raises the question as to what is the nature of these Sūtras such that they could cover all of Mathematics from the book Vedic Mathematics by, ŚRĪ BHĀRATĪ KR□□□A containing sixteen Sūtras and sixteen Sub-Sūtras. Here, however it is taken as hypothesis worthy investigation. In this article examples of Sūtras are given which applying various ways but unify topics because the mental patterns involved are the same. When engaged in mental process the psychology of mathematics involves recognizing patterns of thinking.

As per Tīrthajī's interpretation of Veda, one meaning is the collection of ancient text containing spiritual, as well as practical and the other is knowledge enlightened by true realization which describes principles of working through which study and development of the subject can progress. Sūtras provide us with a new Paradigm for mathematics.

According to author the scope and potential of Vedic Sūtras can be compared with orchestra in which a restricted number of harmonious notes are able to create an immense diversity of melodious symphony. Unfortunately Tīrthajī left for his heavenly abode and we missed out on guidance and additional insights into his work. That is why the extent of applications of Sūtras are unknown and still to be discovered. Much of mathematics concerns finding patterns. As per Poincare views “Mathematics is the art of giving the same name to different things”. When one looks at mathematics in this manner, it becomes a movement dissenting variety on the way to unanimity. Sūtras are concrete reasoning statement containing the experiential foundations which one feels at an individual stage. The most frequently quoted Sūtras in Vedic Mathematics is Parāvartya Yojayet which means Transpose and apply and Transpose and Adjust which has numerical applications. Vedic systems displays the within the processes and also gives an insight into the various quantitative design patterns that become the base of the formulas or the Sūtras. A number of formulae in the Vedic mathematics are expressions of the basic associations of ratio and, in fact they all come under the Ānurūpye Śūnyamanyat Sūtra means Proportionality. One may not find an exact relationship between the particulars of the formula, the various formulas are nevertheless interconnected as they show a similarity in structure. This resemblance and similarity in this structure is the defining property of VM which can be seen in Vedic Gu□aka – Samuccaya□ Sūtra.

A detailed list mentioning applications of Gu□itaSamuccaya□ Sūtra is given in the paper in which all with a common form and may be described in a general way as the process on the aggregate is equal to the aggregate of the process. Intuition is an invaluable tool to solve problems frequently used by Mathematicians. The Vedic Vilokanam means by seeing or spotting and by critically examining the solution is found just by observation. In Vedic mathematics the inherent and the innate method is of equal importance as logical application. Natural process applies beyond quantitative studies. It just tells us that the Sūtras communicate more about the processes of the brain and intuitively picks up patterns of conscious thinking in innumerable circumstances. Some Sūtras even appear to have a deeper meaning than just describing a mathematical formula or processes.

As for example Vedic Vya□□isama□□i□ Sūtra is used for determining middle term from the given terms. In it the individual reflects the whole. The author has concluded for us that it requires to distinguish between the shapes of nature and the seeds of nature. This brings about a perception of the fundamental oneness through which the logical mathematical process and the intuitive understanding can help one to enjoy display of the pragmatic actions and experience magic in our midst.

Thus the synchronization appearing in the world around us is not entirely separate from human intellect. Through this we may come to understand more fully Sri Tīrtha’s overarching statement concerning their universal jurisdiction.

Goyel Pankhuri et al. (2015)

In their paper **“High speed design of Ekanyūnena Pūrve□a Vedic multiplier”, t**he authors have compared the delay obtained in Ekanyūnena Pūrve□a multiplier and array multiplier in case of 4 - bit and other higher ones. The structure of array multiplier comprises of two different types of algorithms. For multiplier of length N, N-1 adders are required. For partial product AND gates are used. In this paper method of Vedic Ekanyūnena Pūrve□a is explained for 2 digit multiplication. Authors have taken as example: 37 × 99.

Here, multiplicand is $37 = 10 \times 3 + 7$.

Comparing it by $10x + y$ we get $x = 3$ and $y = 7$.

Answer is obtained as follows:

The place of thousand is filled as $x = 3$,

The place of hundred is filled by $y - 1$ i.e $7 - 1 = 6$,

Also 10th and unit place is $100 - 37 = 63$.

Thus, $37 \times 99 = 3663$.

This Sūtra is also useful for multiplier 9, 99, 999 and so on. Then they applied this method for higher bits Ekanyūnena Pūrve□a multiplier. Input & output waveforms was obtained in Xilinx 14.2.An evaluation of both Vedic& traditional multipliers is tabulated on the basis of delay in generating final result.

Result shows that 4 and 8 VM works 19.01% and 51.26% faster than their respective traditional multipliers. Less number of LUTS shows that the area required by the Vedic multiplier is less as compared to array multiplier.

Gupta V.K. (2015)

In research paper **“Vedic mathematics and the mathematics of Vedic Period- an Analysis and Application”** the writer describes about the founder of Vedic Mathematics Śrī Swāmī Bhāratī Kr□□□a TĪRTHAJĪ MĀHĀRĀJA who has discovered the Vedic formulae out of deep meditation. Vedic Mathematics is based on sixteen mathematical formula and each formula deals with different branch of mathematics. Vedic mathematics is an approach which exploits both halves of the brain by using the pattern recognition capabilities of one and the analytical capabilities of the other. In this paper, the author has mentioned chapter wise contents of the Swāmijī's book which demonstrate the sixteen formulas. The traditional computation techniques have been progress to include operations such as cubing and exponentiation which not even a computer has mastery over. Further the Ūrdhva-Tiryagbhyām computation procedure provides solutions of linear expressions is very beneficial even for solving 8 or 10 unknowns using the pivoting techniques. Author has explained the use of Vedic Sūtras in simultaneous linear equation, nonlinear differential equations, Integral differential equations, Partial fraction, Trigonometry and Co-ordinate geometry by taking examples of each. At last, writer has concluded that the Vedic approach has many special methods, indeed many more than conventional method.

Even a little exposure of the Vedic Mathematics alongside basic exercise explicitly reveals a novel and radical method of thought.

Gurjit Kaur, Harsimranjit Kaur (2012)

The objectives of their research paper **"Effect Of Vedic Mathematics on Achievement In Mathematics Among Fifth Grade Students"** aim to find out the effects of VM training leading to an increased quantitative ability in fifth standard schoolchildren and also exploring whether such an exposure brings about any improvement in their scientific abilities which obviously operate at diverse levels and also carries out a comparative analysis among boys and girls.

After finding difference between their mean achievements scores they came to the conclusion that the group which had a VM exposure had a superior average accomplishment in tally as opposed to the other group which was trained the old-fashioned mode. It also found that there was no effect on the group in relation to the sex of the participants.

Iyer Karthikeyan (2007)

In a Research paper **"The Vedic Inventive Principles"**, reveals that Vedic Sūtras which are a set of Mathematical Principles are generic in nature and provide seven directions of thought Observation, Division, Equation, Addition, Subtraction, Variation and Rotation. The principles can be used to provide a rich set of thought directions which are suitable to be used during all stages from understanding a problem to creation of ideas and solutions.

Writer has explained which Vedic Sūtras in what ways are generic in nature and provide which directions of thought by giving real life example so nicely.

Writer concluded that the use of triggers or direction of thought provided by the Vedic inventive principles is a well-documented beneficial practice in context of ideation, brainstorming and problem solving.

Jain Dheeraj, Somkuwar Ajay (2013)

In research paper **“Implementation and Performance Analysis of a Vedic Multiplier Using Tanner EDA Tool”** the writers present the detailed procedure to increase the effectiveness of VM multipliers. The above digital multiplier can be executed by Veda inspired mathematics algorithm especially Ūrdhva-Tiryagbhyām Sūtra for multiplications to reduce the intermediate stages of calculations. In this paper, comparative study of different multiplier is done in order to reap the benefits of minimum input requirement & minimum delay.

8-bit Vedic multiplier = 4 (4 - bit Vedic multipliers) + 3 (16-bit adders)

The multiplier blocks are simulated and the outcomes were validated.

The authors discussed an efficient VM layout having real life usage. The applications of this innovative & efficient algorithm ranges from ease of computations to superfast information processing.

Jain Shivangi, Jagtap Vandana (2014)

In research paper **“Vedic Mathematics in Computer: A Survey”** according to writers in the day to day world technology we are using Vedic Sūtras knowingly or unknowingly. Many multipliers are used in computer unit for the multiplication. Multiplier architecture based traditional methods like Array, Booth, Wallace tree etc. are not very efficient when considering the parameters of time and accuracy. Vedic methods were demonstrated to be more efficient than traditional method.

In this paper they have taken a review of papers in which different designs of multipliers based on Vedic Sūtras are taken. They have studied research papers like binary algorithm and de-convolution algorithm which is implemented on the VLSI architecture by using Ūrdhva -

Tiryagbhyām Sūtra and Nikhila□ Sūtra for the division operation, The multiplier used in MAC based on Ūrhva -Tiryagbhyām Sūtra, implementing the multiplication in MATLAB in DSP by using Ūrdhva-Tiryagbhyām, efficient architecture to achieve high speed cryptography algorithm in secure transaction by Advanced Encryption Standard by applying Sūtra, to design high speed energy efficient ALU to reduce the load of processor by using Vedic multiplier Ūrdhva-Tiryagbhyām.

To implement the above mentioned concepts in RSA cryptography applying Vedic algorithm Dhwaja□ka, recent possibility of a speedy coded division (binary) done by the application of Nikhila□ Sūtra and Parāvartya Yojayet Sūtra etc. The writers have written the obtained results from all studied research papers in table form and concluded that Vedic Mathematics Sūtras are useful in different applications like Digital Signal Processing, Image processing, in place of different arithmetic operations like multiplication and Division.

Vedic Multipliers will have a lot of scope in computer field.

Jan HogendikIn (Dec 2004)

The article "**Vedic Mathematics & calculations of Tīrthajī"** has been created in order to respond to various publications in Europe describing the approach of quick oral computations as represented as Vedic maths. In this article writer has given two instances demonstrating the authenticity of Vedic methods. The decimal position and fraction system provided an interconnectivity to the development of techniques of doing calculations quickly which can be attributed to the 20th century Tīrthajī.

Joshi M.P. et al. (2017)

In research paper **"FPGA Based Multiplier"**, the writers proposed an improvement in design of FPGA in areas of optimal operation, sufficient output & well planned impact layout of VM. In FIR filers, microprocessors, DSP and communication application, multipliers are used. Very High Description Language (VHDL) and Very High Speed Integrated Circuits (VHSIC) are also described. For software part, VHDL is used. The applications of VHDL mainly includes formulation, validation & storage of big digital layout. This qualities lead to desire results with minimum efforts.

Section II explains Vedic Mathematics Sūtras.

In section III, as per the recommended design product of two numbers (4 bits) can be obtained by dividing the given number in two halves & then by using 2 × 2 module. The architecture of VM has a similar construction as that of array multiplier and using Vedic Mathematics technique, Vedic multiplier is designed efficiently.

Section IV explains line diagram and algorithm and section V shows the simulation results.

K.Chander Hari (1999)

In research paper **"A critical study of vedic mathematics of Shankaracharya shri Bharati Krishna Maharaj"** the writer's attempt is to decipher and demonstrate Swamiji's method of discovery of the Sūtras with the help of a fundamental algebraic principle enunciated originally by ancient Indian Mathematician such as Bhaskaracarya II. Writer's perspective related to Sūtras is that almost all Sūtras are descriptive of the arithmetical or the algebraic process involved. Neither the Sūtras represent a mathematical theorem nor a new technique like the logarithmic or the trigonometric functions. Erroneous use of Katapayadi in the fabrication of certain Up-Sūtras and the sloka giving the value of π/10 to 32 decimal places provide concrete evidence towards refuting the Vedic origin propounded by Swāmijī.

Kavita, Goyal Umesh (2013)

In research paper **"Performance analysis of various vedic techniques for multiplication"** the writers describe the product operation by old Vedic methods of prehistoric India which in turn are built upon Sūtras. They have compared the timings of all the three techniques Ūrdhva-Tiryagbhyām Sūtra, Nikhila□ Sūtra and Karatsuba –Ofman for both 8 bit and 18 bit numbers. They have concluded that for 8 × 8 bit fastest technique is by using Ūrdhva- Tiryagbhyām Sūtra and for 16 ×16 bit fastest technique is Nikhila□ Sūtra.

Khairnar S. et al. (2012)

In research paper "Vedic Mathematics–The cosmic software for implementation of fast algorithm" the authors provide an overview of the Vedic Mathematics along with other related advanced topics. The algorithm presented finds applications in fields ranging from computing to DSP. In the research publication Ūrdhva-Tiryagbhyām Sūtra for multiplication algorithm is explained by taking example and also Nikhila□ Sūtra for product of numbers in multiple of ten is explained by taking example and it can be modified for binary arithmetic. Uses of these Sūtras are elaborated in the research paper. Performance analysis of Vedic algorithm w.r.t. the parameters leading to minimum input as well as quick and accurate output.

After evaluating the delay factor for the product operation as applied to several logarithms comparing Vedic with traditional approach. The multiplier calculated by Vedic approach was found to be of higher efficiency with reference to the parameters in relation to high speed & less complexity.

Kharel Anshul et al. (2014)

In research paper, **"ALU using Area Optimized Vedic Multiplier"** the writers describe that ALU is the most important part of CPU so there is a need of fast and efficient ALU. Vedic maths is not only restricted to a bunch of tricks and shortcuts for arithmetic computations but it is a way of thinking which can be applied to solve complicated problems of any field relying just on 16 basic Sūtras and 13 Sub-Sūtras. The proposed work is using Vedic Ūrdhva-Tiryagbhyām Sūtra for multiplication which is useful in elevating the productivity rate of calculations part and is also helpful across DSP anywhere. This path breaking work has designed an improved 4 by 4 bit multiplier displaying improved results when comparing available Vedic multipliers. It also reduces the indirect energy supplies. This makes multipliers energy efficient that are extremely beneficial in the long run.

Kothule Sonali et.al (2016)

In research paper **"A Review on Vedic Multiplier using Reversible Logic Gate"** writers have surveyed on Vedic multiplier using reversible logic gate which has wide applications in developing technologies such as quantum computing, optical computing, Nano-technology and so on. Since multipliers have large area, long latency and devour huge energy, suitable multiplier structure will increase the performance and overall efficiency of the system. Reversible logic gate having capability of reversible calculation that means no data loss take place, it can be utilized for decreasing the input requirement. RG comprises of two parts Iv & Ov. Reversible circuit has direct interrelation with Iv & Ov. This vital design constraints have a distinctive advantage that they do not permit fan-outs. Other advantages include least efforts & smallest gate levels (depth).This lower gate levels help optimizing the architecture by generating smallest amount of faulty results. Authors have explained three basic Reversible Logic Gates: 2*2

Feynman gate, 3*3 Peres Gate and HNG Gate with its figures. They have also explained Vedic Ūrdhva-Tiryagbhyām by taking examples of 3 × 3 digit Decimal multiplication and 3-bit Binary multiplication and then Reversible Logic implementation of 2 × 2 Ūrdhva-Tiryagbhyām multiplier. The architectural structure of 4 × 4 Ūrdhva-Tiryagbhyām multiplier circuit in which there are four 2 × 2 Ūrdhva-Tiryagbhyām multiplier unit and three 4 bit binary adder.

As per study, the authors have concluded that the speed of multiplier is increases by using Ūrdhva-Tiryagbhyām multiplier and the negligible power can be achieved by Reversible Logic Gate.

Kumari Sharmila (2014)

Her research paper **"Advantages and benefits of Vedic Mathematics"** describes many pros & cons of VM approach reference to the traditional approach. This approach can be best explained as a way of thinking that involves solving the given cumbersome computations e.g. trigonometry, calculus etc. with the help of basic formulas and set procedures. The development of VM approach has emerged because of eight years research done by TĪRTHAJĪ MĀHĀRĀJA built upon Sūtra and sub-Sūtra in form of word formula. The most significant feature of VM approach is production of swift & accurate results bypassing intermediate lengthy steps. It is a boon for aspirants of prestigious aptitude tests whose questions are designed to test a person's presence of mind & logical reasoning capability under time limit pressure. It also helps in improving confidence and increasing concentration. By using the above systematic approach, most complicated problems are reduced to a simplified trivial problem of arithmetic computation.

By practicing Vedic Mathematics one can reduce Maths phobia and make the subject interesting.

Leonard Gibson et al. (2010)

Their research paper "**VLSI implementation of high speed DSP algorithms using Vedic maths"** is devoted to the design of various N×N multipliers, which uses Vedic Mathematics Ūrdhva-Tiryagbhyām Sūtra. In this paper the authors have used various Vedic Multiplication techniques and methods including Sūtras like Ūrdhva-Tiryagbhyām, Nikhila□ and Ānurūpye□aSunyamanyat for arithmetic multiplications and analyzed it thoroughly in order to find the product of any two numbers. Authors have concluded that Vedic multiplication by

Ūrdhva-Tiryagbhyām showed the least amount of delay as compared to all the types of methods and also reduces the area and speeds up the computation, thus giving the highest efficiency. It reduces the area by using less number of logic elements and speeds up the computation by using short cut method.

Lomte Rashmi, Bhaskar P.C (2011)

In research paper **“Speedy Deconvolution using Vedic Mathematics”,** the writers explain the computing method of linear convolution of two finite length sequence in the same way as polynomial & long hand Divisions are calculated. This research paper is divided section wise. Introduction of novel method for deconvolution is explained in Section 2. In that method, for implementing speedy deconvolution divisor, multiplier and adder used in design must be speedy. Division algorithm is described in Section 3. About Vedic Mathematics and Multiplication method by using Vedic Ūrdhva-Tiryagbhyām algorithm is explained in Section 4. His Holiness SWĀMĪ ŚRĪ BHĀRATĪ KR□□□A TĪRTHAJĪ MĀHĀRĀJA constructed 16 Sūtras and 13 Sub-Sūtras. Urdhva –Tiryagbhyam is one of the Sūtras used for multiplication. A designed multiplier must be compact, consume least amount of input and ensure quick operation. In the digital hardware after comparison of both array and booth multiplier (4 bit)**,** it is found that array multiplier is superior to Booth. The paper proposed algorithm for multiplication by using Ūrdhva-Tiryagbhyām Sūtra. After comparing Array multiplier and Vedic multiplier in Sparten 3E and Virtex 5 Vedic multiplier uses less area and less power.

Thus author has achieved aim of high speed deconvolver implementation.

M. Aruna, G. Usharani (2014)

In research paper **“Simulation & implementation of complex multiplier using Vedic maths”** the writers present the design of a type of multiplier that involves complex numbers. The quickest and the most efficient multiplier is based on Ūrdhva-Tiryagbhyām Sūtra. It can also be used to create the design of a digital multiplier in the same way as the popular array multiplier architecture. However this Sūtra found its initial application in the decimal numbering structure. For multiplication of large numbers Vedic Nikhila□ Sūtra is discussed with example. In this paper mathematical representation of Ūrdhva-Tiryagbhyām and Nikhila□ Sūtra. Hardware implementation of Vedic multiplier which consist of five major segments RSU, Subtraction, adder/subtraction, multiplier & shifter and also mathematical representation of RSU and hardware representation of RSU, Implementation method of complex multiplier is explained briefly. The appropriate test cases have been identified in order to test this modelled complex multiplier design concentrating for reducing the propagation delay.

They have obtained simulation results and concluded that Vedic complex type of multiplier is advantageous, implementable and has high speed as compared to the traditional ones.

Modisane Stephen (Nov 2013)

In the research article **"Calculating powers near a base number**" the researcher has tried to get one line method for calculating digits to the 3rd, 4th, 5th, 6th and 7th power. He has initiated a search for the mathematical truth after Vedic mathematics sixteen Sūtras were disclosed by Jain mathemagic of Australia in 2006.After learning a method of finding square of a number using a one line Vedic mathematics method through intensive inspection by using magic number 9 and the base of 10 writer has discovered a method of calculating a number raised to 3rd power. In the year 2013, during his hospitalization period, writer has discovered a method of calculating a number raised to fourth power and then he has also discovered a method of calculating a number raised to 5th, 6th and 7th power which he has explained in this paper by giving the examples, if a number is less than base of 10 and if a number is more than base of 10 so nicely. The researcher has used Trachtenberg Speed System to confirm his answer and his aim is to develop a method of calculating a number raised to nth power.

Modugu Vasmsi Krishna et al. (2015)

In their research paper **"High speed Vedic Mathematics Multiplier using Compressors"** they have discussed about Vedic Mathematics multiplier Ūrdhva-Tiryagbhyām. Vedic Mathematics provides some algorithms that evaluates fast. Reversible logic plays an important role for continuous reduction of Power dissipation. The multiplier has application over FFT architecture as well as for wireless communications and imaging. High speed arithmetic operations are very important in many signal processing applications.

Mohammed H. Ali & Anil K. Sahani (2013)

In their research paper **"Study, implementation & comparison of different multipliers based on Array, KCM & vedic maths using EDA tools"** they have presented a flowchart of several multipliers including constant coefficient, array and Vedic multiplications. Most applications have basic necessity of minimum input and rare delays. Multipliers characterized by fast operation are particularly needed since out of all the algorithm of DSP, the period of implementation for multiplication is the most dominant. The evaluation of all the multipliers was done after gathering information about their leading parameters.

After comparison of all multipliers results show that the multiplier using Vedic Ūrdhva-Tiryagbhyām Sūtra is the quickest among all.

Muehlman John (1998)

In his research paper **"Maharishi's Vedic maths in elementary education: knowingness to improve affect, achievement & mental calculation"** he has compared Vedic Sūtra based multiplication and checking to traditional method of solving problem at the third grade level students. The results indicates that (i) Those students who used Vedic Mathematics Sūtras scored higher achievement than the students who used traditional method of teaching.(ii) Those students who used Vedic Mathematics Sūtras retained more multiplication and checking skill than the students who used traditional method of teaching.(iii)Those students who used Vedic Mathematics Sūtras enjoyed computation more than the students who used traditional method of teaching.(iv) Those students who used Vedic Mathematics Sūtras computed more efficiently and performed more mental calculation than the students who used traditional method of teaching. After all students involved in the study, structure interviews were conducted.

Author had learnt that students who used Vedic Mathematics found computation easier, more enjoyable and more motivating than the students who used traditional method.

Nataraj Mala Sarwathy, Thomas Michel,

In their research paper **"Expansion of binomials and Factorization of Quadratic expressions"** is a study for validation of the proposition that the computational ability of a person can improve to a great extent by development of psychological models of pattern identification & mapping. In order to test this supposition authors selected some students & explained to them concepts of binomial multiplication via both Vedic & conventional approach. They were then subjected to an assessment based on two time based tests. In first questionnaire, questions included were product of two numbers, product of binomial expressions, factorization of quadratic expressions & expansion of expressions in a real word situation. The another questionnaire, questions included were on application of Vedic techniques for finding product of two numbers, product of binomials as well as expansion of binomials & factorization of Expressions.

The results of the survey did not give any clear indication on superiority among the two approaches. On one hand the VM approach was an addition to the student's collection of tools for solving any algebraic question whereas on the other hand mastering these techniques would require significant efforts. However, the authors are pretty confident that the advantages far outweigh the initial hindrance of formal instruction.

Nicholas A. et al. (2010)

In their book **Applications of VM Sūtra– "vertically and crosswise"** they have explained the application of Vertically & Crosswise Sūtra to solve multiplication, division, squaring and square roots etc. of big digit numbers in a few seconds within few lines of operations.

Arithmetic operations can be greatly simplified by the use of the Vinculum. How to expand the determinant is explained in a shortcut method by using Vedic Sūtra. How to reduce the order of a determinant by one i.e. method of Pivotal Reduction is explained. They also explained evaluating determinant by row and column operations with clarity during working by a spacious layout and orderly (clockwise) sequence for replacing eliminated or modified elements. An alternate solution of expansion of a determinant by extraction of elements (expansion by individual elements) is explained very nicely. Other so many topics like method of finding inversion of Matrices are included.

Nolasco Pietro (Nov 2016)

His research article **"Multiplying numbers nearing multiples of Different bases"** presents an approach to extend the William's method to find product of numbers close to multiples of different bases. An application of Nikhila□ Sūtra was shown by multiplying two numbers which is a power of 10 (10,100, 1000 etc.).

An extension of the method TĪRTHAJĪ MĀHĀRĀJA has been processed by K.R. William to find how to multiply numbers near different bases. This article applies the same pattern of the

William's method considering two values which represents how many times the bases are multiples of the power of ten bases.

Method is explained by multiplication of numbers 3004 and 206 with two different bases 3000 (3 × 1000) and 200(2 × 100). Also they are above their bases by 4 and 6 respectively.

Taking $m_1 = 2$, $m_2 = 3$, $c = 300$, $d = 4$

Left hand side	**Middle**	**Right hand side**
$C \times m_1 + a \times m_2$	$d \times m_1$	$a \times b$
$300 \times 2 + 06 \times 3$	04×2	06×04
618	8	24

Ans. 618824

In RHS two digit is used because base is multiple of 100. On the middle side one digit used because d is one unit. In case more figure, carry over the previous digit. Algebraic proof is also explained. Author concluded that this method is a quick tool in Vedic Mathematics.

An area of research will be to extend the algorithm of multiplication of three and more numbers near multiples of different bases.

P. Maunika et al. (2015)

In research paper **"High Speed Reconfigurable FFT Processor Using Ūrdhva-Tiryagbhyām"** the writers have described a critical block in all multicarrier systems of the FFT processor which is initially utilized in fast paced background. Because of the necessity of the compactness, this system has requisite of the FFT with design of consuming lesser power. However, processor with complex multiplier block which consumes highest power**.** FFT processor comprises of three components like two units of Memory & Butterfly control and third is address generator. The operation of FFT whose butterfly size is r, it is defined as radix r FFT. Author has presented a smart & effective structure of rendering of Redix–four ,sixty four point mainframe of Fast Fourier Transform which avoids modular summation & because of that it boasts the shortest critical route among traditional Radix – four and pipelined framework.

The paper describes a new design of FFT by using Ūrdhva-Tiryagbhyām Sūtra to understand the address generation for Radix 4 which can be extended for the generation of the radix 4^3 which can be combined with structure to calculate or the FFT containing greater range.
Future research points to accomplishments of creating a modularized scalable structure consisting multifaceted points starting from 4k to 64 k, 256 k detailed nodes.

Patwarthan Sushma, Kaur Harjeet (2015)

In research paper **"High Speed System Using Vedic Mathematics-A Survey",** the researchers have discussed about the Multipliers which is key component of DSP, microprocessor & filters (FIR), which are essentially systems characterized by optimum results. They have surveyed six research papers related to their topics and they have concluded that Vedic Mathematics Sūtras are used in different arithmetic operations like division, multiplication, square and cube used in applications like Digital signal processing, cryptography and computation of large calculation.
More research and study in the field of Vedic mathematics and their Sūtras in multiplier will give better results and this has a lot of scope in different fields.

Pawar Roshani, Shriramwar S. (2017)

In research paper **"Review on Multiply-Accumulate Unit"** MAC unit is briefly described and its operation is introduced. The MAC architecture also consists of accumulators along with adders & multipliers. A core objective of the unit is to reduce delay which in turn increase the speed and consume less power. In Digital Signal Processing, speed and throughput are the two parameters and hence there is a need of designing of MAC with enhance speed. The fundamental operation of MAC unit is multiplications. The main concerns in the multiplier unit are latency and also area, speed and power consumption. We choose fast multipliers in various applications of DSP to avoid these issues. In this paper different multipliers are compared with respect to speed and delay namely booth, array & Wallace tree multipliers. Authors have done literature survey on MAC unit by study of few research papers. To design Array multiplier, large number of logic gates are required. A Wallace tree is a digital circuit that multiplies two integers and is one of the efficient hardware implementation. A multiplication algorithm to calculate the product of signed numbers in the notation of compliments of two (binary) is known as booth's algorithm. It is a method that will reduce the number of multiplicand multiples. The performance parameters of MAC units are Power, Clock frequency and throughput. The clock frequency is directly proportional to power factor of MAC unit, therefore reducing clock frequency may proportionally reduce power, on the other hand, the MAC speed and throughput simultaneously reduced. After studied research papers comparing Array, Wallace tree and Booth multiplier, authors have concluded that among them Booth's multiplier is has highest operating speed and consumes less power. MAC unit must be superior when evaluated based on parameters of area, time period and required input, speed and complexity.

Prasad Krishna (2016)

In research paper **"An Empirical Study on Role of Vedic mathematics In Improving the speed of Basic Mathematical Operations",** the author has distributed some basic mathematical calculations, multiplications, square roots, cube roots and subtraction of fractional decimal numbers to a group of 25 competitive examination writing students and told to solve questions without and with using Vedic method techniques. The time taken to complete calculations are taken in minutes and are analyzed by using paired t-test with the null hypothesis that mean score of before and after adopting Vedic mathematics techniques are same. Calculating value of t and then comparing it with five percent LOS from the chart, the above assumption can be ruled out. Thus concluded that Vedic mathematics techniques improve the speed of calculations while solving basic maths problems Limitations and future study.

Pūraṇapūraṇābhyām Sūtra Means: "By the completion or non-completion" to solve cubic and bi-quadratic equations. Calana-Kalanābhyām Sūtra Means: "sequential motion" used to find roots of quadratic equation and also expressions having their degree greater than two. Ekanyūnena Pūrveṇa Sūtra is used in case of multiplication by 99, 999, 9,999…and also for recurring decimal. Ānurūpyeṇa Sub-Sūtra Means: "Proportionality" In order to evaluate the product of numbers which are both closest to a multiple of 10, Ādyamadyenāntyamantyena sub-Sūtra means: "The first by the first and the last by the last".

An application of Yāvadūnam Tavaduni Krtya Varganca Yojayet Sub-Sūtra is for calculation of product of a number with itself near the multiple of ten and also to find cube. An illustration was shown to explain the above application in a detailed manner. For multiplication of the numbers of the type 25 × 25, 23× 67 in which sum of the last digit of both the numbers is 10

Antyayordasake’pi sub-Sūtra is used. Antyayoreva Means only the last term, in solving special type of equation.

Lopanasthāpanābhyām is used for factorization of Quadratic expression of not only the homogeneous nature but also 3 variables generic expressions. For the verification of factorization of quadratic expression, Gu□itasamuccaya□ Samuccayagu□ita□ Sub-Sūtra can be used. Vilokanam Sub-Sūtra Means “by observation”.

Puranmath Somshekhar, Kumar Priyatam (2014)

In research paper **“High Speed Area Efficient Hardware Multiplier (Using Vedic Algorithm)”**, writers present an area efficient Vedic multiplier using the two ancient Nikhila□ Algorithm and Ūrdhva-Tiryagbhyām Algorithm. When data is large, Nikhila□ is faster than Ūrdhva -Tiryagbhyām Sūtra. DSP unit demands for high speed multipliers which use Vedic algorithm like Nikhila□ and Ūrdhva-Tiryagbhyām implemented on FGPA devices like Spartan, Vertix and Altera. Section II explained Vedic algorithm Ūrdhva-Tiryagbhyām for multipliers of two, four & eight plus bits and also explained Nikhila□ Sūtra and its hardware design. In section III the proposed multipliers were designed and implemented using Xilinx. They have compared karatsuba algorithm, optimized Ūrdhva-Tiryagbhyām and Nikhila□ with respect to delay and area utilization.

They have concluded that Vedic algorithm are best suited for high speed applications and simplifies by implementing the proposed design multiplier of complex modules for DSP and image processing.

Puri Narendra, Weinless Michael (1988)

In their research paper **“Vedic maths the cosmic software for the cosmic computer”**, the writers have discussed about a new holistic approach offered by Vedic mathematics to mathematics and to Vedic mathematics education. Each Vedic Sūtra portrays a real process for calculation that can be applied on one level while on a deeper level each Sūtra are seen as fine procedure to obtain hundred percent consistency for active psychological impulses. This leads to quick results of complicated questions. In Maharishi’s word the coherently functioning brain can be compared with Cosmic Computers and VM formulas with Cosmic Software. They also found that for the students the computational method is quick, lucid in comprehension sand more enjoyable than conventional methods.

Puri Narindra (2012)

In the book **“Vedic Mathematics-Methods”** according to writer, entering the method and procedure of **Vedic Mathematics** which consists of Sūtras & Sub-Sūtras in the form of word formula discovered by TĪRTHAJĪ MĀHĀRĀJA (1884-1960), it will be easy to understand the applications of it & to solve different type of problems quickly. Author has started to explain with the Ekādhikena Pūrve□a Sūtra for finding square of a number consisting the unit digit five, vulgar fraction whose denominator ending in nine by division and multiplication method explained with its algebraic proof. Nikhila□ Navataścaramam□ Daśata□ Sūtra means: “All from nine last from 10” which is very useful for the product of two numbers whose multiplier is either less or more than the base, which is in multiple of 10. Nikhila□ Sūtra method for product of two numbers different cases like both the numbers are less than base, both the numbers are higher than base, one number is less than and the other is more than the base is explained with algebraic proof. Nikhila□ Division is also explained. The meaning of Sūtra Ūrdhva-Tiryagbhyām is “Vertically & Crosswise” is appropriate for the product of any types of two numbers and also by using Ūrdhva-Tiryagbhyām Sūtra conversely one can solve problems of the type dividing one big number by the another big number, Parāvartya Yojayet closely related to two well-known theorems for division (one is Remainder & the other one is the procedure of Horner) used for division, in solving equation. Śūnyam Sāmysamuccaye Sūtra is used for verifications of the already obtained results of factorization problems, Ānurūpye Śūnyamanyat is used for special type of simultaneous equation, Sa□kalana-Vyavakalanābhyām Sūtra means: “by addition and by subtraction” for solving those two equations whose second equation is obtained by exchanging the x & y coefficient of the first equation. Author has explained all the Sūtras by taking examples of each with algebraic as well as Vedic solution.

In the end he concluded that the method for solving problems by using Vedic Sūtras are more accurate as well as speedy than algebraic method.

R. Sridevi et al. (2013)

In research paper **“Design of a high speed multiplier (Ancient VM)”,** the writers present an optimal area multiplier design built on algorithm of ancient VM. The multiplier uses Nikhila□ Sūtra for finding product in a way that decreases the amount of rework and thus ensuring quick results. The authors have suggested a multiplier whose design is characterized by compactness, productivity & reliability for the applications of FPGA. In research paper writers have described that Vedic Mathematics which can be completely built from thirteen Sub-Sūtras along with 16 Sūtras has been revived by TĪRTHAJĪ MĀHĀRĀJA. An analogy for the Vedic Sūtras is made with a software that handles the functioning of our cosmos. Vedic for multiplication is explained deeply by giving examples. They compared the results of Vedic multiplier with conventional multiplier. They have concluded that Nikhila□ multiplier is more compact & requires short operational period in comparison to the traditional multipliers. Proposed 4 × 4 multiplier architecture exhibits speed improvement. The above method is more appropriate for evaluating the product of numbers with bit length greater than sixteen.

Rajsekhar, Anbalagan (2016)

In research paper **“Design of High Speed Optimised Vedic Multiplication Techniques”,** the writers have proposed a multiplier by using Ūrdhva-Tiryagbhyām of Vedic mathematics which provides the answer of the critical problems. Using Vedic technique, various ALU were planned & merged to convert them to VM type. Therefore this new unit is well-suited with co-processor. A variety of fundamental & intricate computational problems can be solved easily by VM approach. By using Ūrdhva- Tiryagbhyām multiplication method is explained by a square diagram with the sides represented by the numbers whose product is to be found. As indicated in the above paper, design starts first with basic blocks of 2 by 2 bits multiplier and then by using this block, 4 by 4 bit block is developed and by using 4 by 4 block, 8 by 8 bit block, 16 by 16 & eventually thirty two bit block multipliers has been made and concluded that ALU designed with Vedic overlay high speed multiplier shows swiftness & compactness which in turn contribute to perfect execution.

Ramachandran. S, Pande Kirti (2012)

In research paper **“Design, Implementation & Performance Analysis of an Integrated Vedic Multiplier Architecture”,** the authors have introduced Vedic Mathematics. The main and central character of Vedic Mathematics as explained by the authors is that it is able to make long and complex calculations easier and quicker than those done by traditional mathematical methods. The paper presents Vedic Mathematics Sūtra based efficient algorithm like Ūrdhva-Tiryagbhyām and Nikhila□ Algorithm and says that it finds applications in many divisions of Engineering such as in Digital Signal Processing. This branch of engineering is extremely dependent upon the efficiency of time and by using VM it reduces the multipliers and hence reduces time.

There are three ways of sorting the structure of multipliers: Serial, Parallel and Serial and Parallel. The first multiplier is used serially and works on the optimal use of hardware; while the Parallel one does mathematical processes in higher efficiency. The combination of both i.e. Serial - Parallel Multiplier is the best transaction because it provides best payoff between the two multipliers: by significantly creating a new dynamics between time and area of the earlier multipliers The Research papers were studied by the authors related to creating a super-fast multiplier by the use of VM and Implementation of Vedic algorithm by conducting a relative analysis of the time utilized in the procession of mathematical calculations by using traditional mathematics and comparing it with VM techniques. It found that in terms of efficiency, in terms of space and in terms of usage of electricity VM and VHDL were better. The above paper explicates, with examples, multiplication of two numbers by means of Ūrdhva-Tiryagbhyām & Nikhila□ Sūtra. Also architecture of 64 - bit Urdhva-Tiryakbhyam Sūtra & proposed architecture of Nikhila□ Sūtra based multiplier is presented. After study of related work, the researcher found that Ūrdhva-Tiryagbhyām Sūtra which has its basis on VM works more efficiently in case of extreme figures. Nikhila□ Sūtra performs better. Therefore a construction using Integrated VM design is suggested by the researcher.

On the premise that keeping the original conditions at about twenty percent of the closest base as the limit of Nikhila□ if the input to Ūrdhva-Tiryagbhyām established multiplier will be ‘ON’ i.e. it will perform the multiplication and Nikhila□ based multiplier will be ‘OFF’. The multiplier created on Nikhila□ will work only if the range of the number is within the limit of Nikhila□ I.e. it will perform the multiplication and Ūrdhva-Tiryagbhyām multiplier will be ‘OFF’. The suggested construction aims at getting more efficient outcomes which are also energy efficient.

Simulation results have shown on the basis of the operating conditions the multiplier Sūtras operate only one at a time and never simultaneously. If the numbers are big and nearing the bases of 100, 1000, 10,000, etc. then the multiplication is taken over by Nikhila□ Sūtra and this helps in faster multiplication. With smaller numbers the multiplication is done by the Ūrdhva-Tiryagbhyām multiplier and the Nikhila□ Sūtra remains non-functional. According to the report of speed analysis, in general the Ūrdhva-Tiryagbhyām Sūtra gives a better performance when the multiplier is 8 bit than Nikhila□ Sūtra. This is as the size of the number is small. When the magnitude of the multiplicands becomes large, Nikhila□ achieves better results than Ūrdhva-Tiryagbhyām.

In the end the authors have the concluded that the efficiency and the speed of the process is better with Vedic Mathematical algorithms. They therefore suggest an integrated VM to create more efficient system and Sūtra is to be chosen subject to the size of the input number.

Raman Ashish et al. (2010)

In the paper **"High speed reconfigurable FFT design by Vedic Maths"** the writers have discussed about the Fundamental operation FFT (Fast Fourier Transform) which is typically performed in any DSP (Digital Signal Processing) system. The paper presents the compilation of DSP & techniques of Field programmable gate array (FPGA). Also the contribution of FPGA in DSP simulation has been highlighted. The author discusses a compact FFT prototype based on VM. Ūrdhva-Tiryagbhyām is utilized to improve its efficiency**.** It also describes the basics and the range of applicability of this methodology. After detailed comparative case study the authors have concluded that FPGA employed units are far more superior in terms of swiftness, compactness & accuracy to the traditional units.

Reddy Kishor Kumar, Subbaiah Vankata (2014)

In research paper **"A high speed vedic multiplier using Nikhila□ Sūtra with barrel shifter"**, writers describes the simulation of 64 bits VM with modified design of Nikhila□ Sūtra enriched related to transmission lag with respect to traditional ones such as modified booth, braun & wallace tree multiplier. However, the lag was found to have decreased drastically on application of Barrel Shifter across several stages.

The modified Nikhila□ Sūtra based architecture is explained briefly in section –III.

The equation can be expressed quantitatively in form of variables as belows:

$$P = X * Y = (2^{K_2}) * (X + Z_2 * 2^{(K_1 - K_2)}) + Z_1 * Z_2$$

Where,

K_1= Maximum power index for X

K_2 = Maximum power index for Y

Z_1 = Resudue in X

Z_2= Residue in Y

The components placed in this equation can be divided in 3 types. The authors have explained these component well in detail. Also sufficient illustration have been shown to make the fundamentals clear. The structural layout computes this quantitative formula.

After comparing simulation result of conventional Array multiplier and proposed multiplier in terms of delay, the authors have observed that there is 95% reduction in delay from proposed multiplier to Array multiplier. They have concluded that the increase effectiveness & productivity achieved can be attributed to the inclusion of VM employing Barrel Shifter. According to authors the manipulative capability of any person in all fields is better.

S. Smitha (2017)

In research paper **“Strengthening Critical Thinking Skills Of Prospective Teachers Through Applications of Vedic Mathematics**” the study was conducted to test the success of the highest influence of Vedic Mathematics in reinforcement of Computational Speed and Critical Thinking Ability among the member of prospective teachers and also give them with self-assurance by gaining that skill of Computational Speed and Critical Thinking Ability through tradition of Vedic.

Under the hypotheses that Vedic Mathematics applications are very much effective in strengthening Computational Speed and Critical Thinking Ability among the member and Achievement of the skills of Vedic Mathematics applications is a true solution in equipping prospective teachers with adequate skills. The sample of 100 students from various disciplines of one training college of Kerala were selected using random sampling technique. Study was undertaken for one week.

The tools used for the study were modules prepared on Vedic Sūtras like Ekādhikena Pūrveṇa Sūtra, Ekanyūnena Pūrveṇa Sūtra, Ūrdhva-Tiryagbhyām Sūtra & Nikhila□ Navataścaramam□

Dastah and also Computational Speed Test and Critical Thinking Ability test. Experimental and control both the groups were divided into 50 team members as the mathematics teacher. After the data analysis and discussion she found that her hypotheses is true. Then she concluded that teachers' training has central significance and will help in spreading the Vedic Mathematics techniques to the next cohort of students. Therefore, the edification of teacher has to keep up-to-date, ever-evolving and dynamic.

S. Sujatha, V.P.Krishnammal (2016)

In research paper "**Performance Analysis of Ānurūpye Vedic Multiplier in FFT Processor"** writers have introduced that the revival of Vedic Mathematics which based on Sixteen Sūtras can be traced to Atharvaveda in the period 1911and 1981 with the perseverance of Śrī Bhāratī Kr□□□a. Algorithm by using ancient methods is applied to various branch of Engineering.

In this paper by using Vedic multiplier with Ānurūpye Śūnyamanyat Sūtra (i. e. If one is in the ratio the other is zero) FFT processor is designed, to compute N-point which provides a fast and reliable approach. Because of the Ānurūpye□a sub-Sūtra based on Proportionality the intricacies of layout decrease with input of high bit numbers & superior integrity. Three-digit multiplication (e.g. 532 ×438) by using Ānurūpye□a Sub-Sūtra is explained in this paper.

In this method first select a theoretical base which is common to both numbers (here base is 100) and then take a proportion

(Here 100 × 5 = 500).

5 3 2 +32
× 4 3 8 -62
4 7 0 -1984

First take deviation of multiplicand and multiplier from 500 and write it in the corresponding line in RHS. Then multiply RHS, we get answer -1984. Since 100 is the theoretical base, write two digit in RHS (i.e. 84) and -19 carry over.

In the LHS (calculating 532 - 62 = 438 + 32 = 470) write 470.Since working base is 500, multiply 470 by 5 and then in the result add carry.

L.H.S is = 2331 (470 × 5 = 2350 & 2350 -19 = 2331).

R.H.S. is 84. 100 - 84 = 16 and subtracting 1 from 2331 i.e. 2331 - 1 = 2330.

Thus, final answer is 532 × 438 = 233016.

Ānurūpyeṇa Sub-Sūtra is applied to plan the structure of Vedic multiplier of type 8 bit and simulation result is compared with Ūrdhva-Tiryagbhyām Sūtra. It shows that Ānurūpyeṇa multiplier gives optimum performance than Ūrdhva-Tiryagbhyām in terms of reduced delay and power. Implemented FFT processor using both Vedic Ūrdhva-Tiryagbhyām Sūtra and Ānurūpyeṇa is compared with Array multiplier.

The simulated result with respect to Power, No. of gates & time lag shows Ānurūpyeṇa based multiplier gives more efficient performance than Array multiplier.

Sapkal Krutika, Sharwankar Urmila (2016)

In their research paper **"Solving Linear Equation Using Vedic Mathematical Approach on Parallel Platform"** The main purpose of the writers Sapkal Kritika, Sharwankar Urmila is to survey and form a comparative analysis of all the existing techniques to solve linear equation on parallel platform. They have surveyed methods for solving linear equations irrespective of the platform, methods for parallel platform and analysis of performance. After further analysis of the previous works they have noted that different conventional methods of solving linear equations such as Gaussian elimination, Cholesky factorization, LU factorization, QR factorization fail to give the desired faster and accurate result after certain level. They have also approached the linear equation solving in Vedic Mathematics based on the various Sūtras like the Parāvartya Yojayet, Ānurūpye Śūnyamanyat, Saṅkalana-Vyavakalanabhyam & Sopāntyadvayamantyam Sūtra and they have found that Parāvartya Yojayet Sūtra is the best fit for dealing with linear equations on a parallel platform.

Vedic Mathematics provides a very different outlook towards the approach of solving linear equations, on parallel platforms, for reducing space consumption and minimizing the number of algebraic operations involved in solving linear equation.

Sengupta Diganta, Sultana Mahamuda, Chaudhuri Atal (2013)

In their research paper **"Vedic division- A fast BCD division algorithm facilitated by vedic mathematics"** the writers have compared Vedic Division algorithm for BCD division with conventional algorithm in literature. Out of SRT division algorithm, Restore and without restore type division algorithm last one is fastest and economic. But considering the requirement of parameters speed and memory, the proposed algorithm performs better w.r.t. it. The division algorithm without restore type time approximation and division both are in proportion with the no. of bit. While in above proposed one computation has to be done upon digits rather than on bits and compute correctly up to 38 digit which is equivalent to one hundred and twenty seven bit. By modifying it can be divided even larger numbers. They have explained the algorithm of the best case and worst case examples of division where the estimate time is dependent upon the no. of remainder normalization required. Proposed Vedic division algorithm expresses the study of completion of time by Vedic-division and without restore type division in table form concluded that in Non-Restore algorithm execution time increases with increasing no. of bits and the length of the division is depend upon how big the quotient is. In proposed algorithm time required is constant which is not depend on the dividend and nor on the length of the division as well as how big the quotient is as it depends upon the requirement of no. of remainder normalization.

Further, VLSI implementation of the algorithm remains to be tested.

Sharma Gaurav et al. (2013)

His research paper **"Delay comparison of 4 × 4 vedic multiplier based on different adder architectures using VHDL"**evaluates multipliers (four bit) of two types- one with ripple carry adder and another with carry look ahead adder of Ūrdhva-Tiryagbhyām based 4 × 4 multiplier architecture can be executed with the help of multiplier units of 2 × 2 type. A comparison of the two types of multipliers was made focusing particularly on the time consumption and swiftness aspects of each. The results of the study were tabulated and inferences were drawn. This data highlighted the advantages & disadvantages of both types of adders.

An amalgamation effect proves that for 4 × 4 VM, the second type consumes lesser time period and thus has higher speed than the first Ūrdhva-Tiryagbhyām type.

Shriki Atara (2014)

In the article **"Looking at Algebraic Expressions through The lens of Vedic Maths",** the author came across some You Tube short videos presenting the ancient mathematics of India – Vedic mathematics. Vedic methods are easy to understand. This article suggests exposing students to the beauty of Vedic Mathematics through an introduction with simple arithmetic rules, along with practicing the manipulation by multiplying two-term polynomials. This article includes four activities demonstrating Vedic methods for multiplying special numbers that are close to a power of 10, 2-digit numbers having the same tens digits and the sum of their units digits in 10, 2-digit numbers having the same units digits and the sum of their tens digits in 10 and squaring 2-digit numbers. The activities included in this article aim to expose students to Vedic Mathematics; to provide ideas for breaking the routine practice of basic algebra, in particular the multiplication of two-term polynomials. These four activities reveal only 'the tip of the iceberg' of many opportunities to expose students to Vedic Mathematics, and to enable them to practice basic algebraic manipulation through understanding their meaning.

Singh Anuradha, Gupta Neetu (2017)

In their research paper, **"Vedic Mathematics for VLSI Design: A review",** they have explained different Vedic Mathematic techniques for multiplication like Nikhila□, Ūrdhva-Tiryagbhyām, Ānurūpye□a which are fast, accurate and less area is consumed compared to traditional methods and also for binary division Vedic Sūtras like □, Nikhila□ & Dhwaja□ka. All these Vedic Sūtras with examples are explained very nicely in Section – II. In section – III algorithm of Vedic multiplication for two bit number and then extended for highest bit number. In this section research work done by different researcher related topics like proposed 32 × 32 bit number based on Urdhv-Tiryagbhyam Sūtra, Convolution, DSP algorithms, High speed Karatsuba Multiplier, High speed Reconfigurable FFT using Vedic Sūtra, Vedic Multiplier, Vedic binary division by Vedic Sūtras Parāvartya & Nikhila□, Floating point division by Nikhila□, RSA Cryptography in implementing RSA encryption and decryption system by using Vedic Algorithm resulting briefly.

After study of some research papers they have concluded that employing these Sūtras in the computation algorithm of digital system complexity of architecture, execution time, area and power consumption.

There is further scope of developing efficient and faster algorithm based on Vedic Sūtras.

Singh Vikram et al. (2015)

In their research paper **"Implementation of vedic maths Sūtras & barrel shifter in designing of multipliers",** the writers have explained about the important part of ALU of a Processor Barrel Shifter and Multiplexer which is considered as a foundation to the advanced theory. This paper presents a comparative study among different multiplier designing approaches is aimed to present the use of Ūrdhva-Tiryagbhyām Sūtra for enhancing the speed of multiplication operation also by using Nikhila□ Sūtra problem may further be more simplified and the number of elements may be reduced.

After the comparative study they concluded that the use of Nikhila□ Sūtra is advantageous.

Syed Azman bin Ismail, Pumadevi (2010)

In their research paper **"Multiplication with the Vedic Method",** the writers have conducted a case study in order to investigate the cause of inability to face basic arithmetic problems among students. The main goal of this study was to develop an easy to understand approach that can help solve the problems involving calculation of product, especially higher numbers. According to the authors students usually have a hard time recollecting product tables above six.

In order to solve the above issue, author has selected 5 students out of 30 students of class of 4th grade and they gave them a questionnaire comprising of basic arithmetic operations. The results of this pre-test were compared with those of post test conducted after explanation of Sūtras of Vedic maths. One of the Sūtra of VM is Vertically & Crosswise Sūtra for multiplication which was found to be very useful. The effectiveness of the Vedic approach was validated when the students gave a positive response in the interview to assess the suitability & applicability of VM in academic course work.

This result can be easily derived from the comparison done based on the observations recorded systematically in tabular format by the author.

Thakre Laxman et.al (2010)

In research paper **"Performance evaluation & synthesis of multiplier used in FFT operation using conventional and Vedic Algorithm"** Vedic algorithm is proposed in order to execute multipliers in FFT and compare it with Vedic algorithm. The best part about VM is that it can reduce even the most complicated expression to a simplified task of arithmetic manipulation. It describes several high performing algorithm which find applications in DSP & FFT. The shortcut prescribed by VM contribute in minimizing the efforts & thus enhancing productivity. Various Sūtras like Ūrdhva-Tiryagbhyām, Nikhila□ Sūtra, Ānurūpyena & Vinculam are useful for their swiftness & accuracy. In Digital Signal Processing Vedic Mathematics algorithm is applied to digital multipliers. By taking example of multiplication of 4 digit number by 4 digit number Ūrdhva-Tiryagbhyām Algorithm is explained and in order to make the same algorithm applicable to digital type of hardware, binary system is preferred. The need for performance evaluation dictates conventional multiplier for 4-bit to be formulated & integrated by 4 bit VM in individual operation. Result of comparison of both types of multipliers (Traditional & VM algorithm) considering the parameters like No. of slice LUTs, used as Logic, No. of IOs, Total memory used, No. of LUT FF pairs and BELs are written in the table form and hardware requirement of all the parameters for conventional and Vedic Multiplier present graphically. It has been observed that the implementation of traditional multiplier requires more than half of the space & bit size.

They have concluded that the proposed technique by using Vedic multiplier has more advantages with respect to the traditional multiplier.

Trivedi Ankit et.al (2017)

In research paper **“Vedic Mathematics for High Speed Multiplier Designs: A Review”** writers have introduced sixteen Sūtras of Vedic Mathematics of which Ūrdhva-Tiryagbhyām is most important for fast multiplication. Several multipliers with swiftness are suggested by Ūrdhva-Tiryagbhyām. In this paper Ūrdhva-Tiryagbhyām Sūtra is explained for two and four digit numbers and then it prolonged for binary numbers. By applying it to 2 × 2 bit multiplication and stretched it till 16× 16 bit & in the same way for Nikhila□.

In section –III authors have review of 40 research paper written on Sūtras Ūrdhva-Tiryagbhyām and Nikhila□ implemented in DSP and in applications of DSP like convolution, in MAC unit and also Reconfigurable FFT design using Vedic multiplier on FPGA, in VLSI for RSA system, curve encryption

After discussing all the papers, the authors concluded that in digital plan multiplier with Vedic Sūtra is most effective with respect to swiftness**,** power & area with respect to further multipliers and the technique by using Vertically & Crosswise Sūtra is more favorable technique in terms of time and area.

Urmila Rani, (2014)

In research paper **"Vedic Mathematics- A controversial origin but a wonderful Discovery",** the researcher has discussed about the wonderful discovery of Vedic Mathematics by giving list of Sūtras and Sub-Sūtras claimed to be discovered from Vedas. According to her, the applications of these Sūtras in the different fields have not been objected, but the critics of Vedic Mathematics object to its origin and name. She has given the names of different authors and their views about the origin of Veda. She has also mentioned applications of Vedic Mathematics in computer science in designing hardware by using Vedic Mathematics Sūtras.

She has concluded that the Vedic Mathematics discovered by Tīrthajī can be used in high school mathematics for speedy calculations.

V. Aiholli, Aneppanavar Jyoti (2012)

In the research paper **"A study on effectiveness of Vedic Mathematics in teaching Mathematics"** the purpose of the study of writers is to know the effectiveness of the Vedic Mathematics versus traditional methods of teaching Mathematics by finding Mean scores of pre-test and post-test of Vedic method of teaching. For finding results authors have employed experimental method by taking 30 students as an experimental group and 30 students as controlled group. They have used Pre-test and Post-test on both group as a tool.

It is found that effectiveness of Vedic Mathematics of teaching is more than that of traditional method of teaching on students.

V. V. Krisna, S. Kumaar (2012)

In research paper **"High speed, power & area efficient algorithms for ALU using vedic maths"**, writers have used Vedic techniques for the structure of algorithm and their execution for ALU which consume less power and is faster. The central aim of this work is to create a blue print for an arithmetic logical unit which can be utilized in all applications of a processor. The efficiency of ALU lies with the way the multipliers function. Many procedures have been created. At the framework stage, multiple methods are proposed & worked out on the feat for a drastic increase and radical rise in effectiveness by the way of condensing the partial products and addition. By using the above mentioned methods, a sharp decline in complications, implementation period, mean energy consumption & overall hardware space required was observed relative to the prevalent computational procedures.

Current trends predict a surge in ubiquity of VLSI powered compact and cascaded integrated circuits active in disruptive areas of research and technological applications demanding challenging constraints of dimensional parameters.

Verma Pushpalata, Mehta K. (2012)

Their research paper "**Implementation of an efficient multiplier based on Vedic mathematics using EDA tool"** presents a structure of a high speed 8 bit VM which is based on a different method of calculation than the traditional one. Many DSP structures as well as regular high speed processors rely to a very large extent on the multiplier because they are central units in any hardware.

Authors have concluded that multiplier that apply the Vedic multiplication method grounded on Vertically & Crosswise Sūtra is better than the methods used in traditional multipliers like Array & Booth.

Vogel Steven (Sept 2011)

In article **"An interesting Application of Vedic Mathematics",** Vedic Mathematics Newsletter Issue No. 76, Sept-2011, Editor: Kenneth William, explains compound interest with examples; one of the best example application of VM in carrying out cumbersome calculations is finding out the compound interest for a given principal over a period of time.

The general formula is:

$FV = PV * (1 + i)^n$

According to author, this consequent computation can be carried out very quickly & easily by Vedic approach.

By explaining how to evaluate $(1.04)^4 = (1 + 0.04)^4$

Let $a = 1$, $b = 4$ and $b/a = 4$.

Each term can be founded by multiplication of previous term and b/a, factors can be founded from the binomial theorem (by using Pascal's Triangle) and product of each term with corresponding factor can be obtained.

William Kenneth (2003)

In his book **“Vedic Math Genius**”, the author has explained Give and Take by using Vedic Sūtras “By Addition and By Subtraction”, “All from nine and last from 10”; Mighty Multiplication in special cases**;** general multiplication method by using Vertically & Crosswise Sūtra; Squaring the number by Duplex method so nicely by giving different type of examples for each. The traditional approach taught academically for solving basic arithmetic operation is unidirectional. I.e. we need to solve the expression systematically in order from L.H.S. to R.H.S. This hinders creativity & ingenuity. On the other hand VM stresses on the fact that there can be multiple ways to approach & solve any given problem. Also how to find square root, “The crowning Gem” by taking division examples and “Math or Magic” by taking examples of how to convert fraction into recurring decimal are explained by using Vedic Ekādhikena Pūrve□a Sūtra and Ānurūpye□a Sub-Sūtra. Finding the roots of expression by using Vedic Parāvartya Yojayet; Quadratic Equation by using calculus Sūtra, Solving Simultaneous Equations by using Ānurūpye Śūnyamanyat Sūtra; for verifying the already obtained result of special type of equations by using Gu□itasamuccaya□ Samccayagu□ita□ Sub-Sūtra. In the end of the book the author has given list of sixteen Sūtras and also an explanation of few Sūtras.

William Kenneth (2017)

In article **“Bharati-Krishnas-Special-Cases”**, the writer describes educational advantages of Vedic Mathematics and three categories in which the special methods can be researched are suggested by giving examples of each. The inclusion of Trigonometric functions and Derivatives of function where numbers or polynomials are used by SWĀMĪ ŚRĪ BHĀRATĪ KR□□□A TĪRTHAJĪ MĀHĀRĀJA. The use of special methods in Vedic Mathematics can be a huge saving in time if we know the problem has a special characteristic, we can solve it by the special process required. After teaching Vedic Mathematics methods students can apply their own decisions and inputs and by explaining the solution of the problem with special cases they can be encouraged to extend and developed the concept to make the subject more creative. In this paper the author has distinguished area of research in three parts: Reconstructed forms which fall within known special types, increasing the series of application of the given special methods of Bhāratī Kr□□□a and finding the area of research which is not touched by him. All the three parts of the area of research are explained with few examples with new applications from the book of TĪRTHAJĪ MĀHĀRĀJA to encourage others for new research in the applications of Vedic Mathematics and use it as an educational tool in schools. According to author this is the way to change the reaction of a child to a question of Mathematics “let me look closely to see what I can find out” instead of “what algorithm am I supposed to apply?”

Chapter 03

Vedic Mathematics in Quadratic, Cubic and Quartic Equations

Quadratic Equations:

The general quadratic equation in variable x containing a, b and c as constants is

$$a.x^2 + b.x + c = 0, a \neq 0.$$

The current method for solving quadratic equation is by using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

by using the above formula, two values of x can be found out.

The quadratic equation can also solved by using Vedic method which is superior to the current method.

3.1 Special type of Quadratic Equations by using Vedic Sūtra:

Quadratics of certain forms are solved using Sūtras more effortlessly and more speedily than the above method. The solution of quadratic equations by using different **Sūtras** and **Sub-Sūtras** are explained below:

3.1.1 First Special Type under Vilokanam Sub-Sūtra:

To solve the equation of the type in which L.H.S. is in the arrangement of either addition or subtraction of the two Reciprocals.

I.e. either $x + \frac{1}{x}$ or $x - \frac{1}{x}$

According to Vilokanam Sub-Sūtra, we can split R.H.S. into the same form.

Solve:

[1]

$$z + \frac{1}{z} = \frac{10}{3}$$

By using current method,

Taking L.C.M. we get,

$3z^2 + 3 = 10z$

$\therefore 3z^2 - z - 9z + 3 = 0$

$\therefore z (3z - 1) - 3 (3z - 1) = 0$

$\therefore (z - 3) (3z - 1) = 0$

$z = 3$ OR $z = 1/3$

$\therefore$ Roots of given Quadratic equation are $z = 3$ OR $z = 1/3$

By using Vilokanam,

Here, the right side term is $\frac{10}{3}$ which can be written in the form of addition of two reciprocals $3 + \frac{1}{3}$

$$\therefore z + \frac{1}{z} = 3 + \frac{1}{3}$$

$\therefore z = 3$ OR $z = \frac{1}{3}$

$\therefore$ Roots of given Quadratic equation are $z = 3$ OR $z = \frac{1}{3}$.

[2]

$$y + \frac{1}{y} = \frac{50}{7}$$

By using Vilokanam,

We observe that the right part of given equation is $\frac{50}{7}$, which we can write in the form of sum of two reciprocals $7 + \frac{1}{7}$

$$\therefore y + \frac{1}{y} = 7 + \frac{1}{7}$$

$\therefore y = 7$ OR $y = \frac{1}{7}$

∴ Roots of given Quadratic equation are y = 7 OR y = $\frac{1}{7}$

[3]

$$\frac{y}{y+1} + \frac{y+1}{y} = \frac{26}{5}$$

By using Vilokanam,

Here, the right term = $\frac{26}{5}$ can be written in the form of two reciprocals $5 + \frac{1}{5}$

$$\therefore \frac{y}{y+1} + \frac{y+1}{y} = 5 + \frac{1}{5}$$

$$\therefore \frac{y}{y+1} = 5 \quad \text{OR} \quad \frac{y}{1+y} = \frac{1}{5}$$

∴ y = 5 + 5y OR 5y = y + 1

∴ y = - $\frac{5}{4}$ OR y = $\frac{1}{4}$ are roots of given quadratic equation.

[4]

$$z + \frac{1}{z} = \frac{25}{12}$$

By using Vilokanam,

Here, the right side term = $\frac{25}{12}$

Which is addition of two reciprocals

$\frac{4}{3} + \frac{3}{4}$

$$\therefore z + \frac{1}{z} = \frac{4}{3} + \frac{3}{4}$$

$$\therefore z = \frac{3}{4} \quad \text{OR} \quad z = \frac{4}{3}$$

∴ Roots of given Quadratic equation arez = $\frac{3}{4}$ OR z = $\frac{4}{3}$

[5]

$$\frac{w+7}{w+3} + \frac{w+3}{w+7} = \frac{53}{14}$$

By using Vilokanam,

Here, the right side term $\frac{53}{14}$ can be written in the form addition of two reciprocals

$\frac{7}{2}+\frac{2}{7}$

$\therefore \frac{w+7}{w+3}+\frac{w+3}{w+7}=\frac{7}{2}+\frac{2}{7}$

$\therefore \frac{w+7}{w+3}=\frac{7}{2}$ OR $\frac{w+7}{w+3}=\frac{2}{7}$

$\therefore$ 2 (w + 7) = 7 (w + 3) OR 7 (w + 7) = 2 (w + 3)

$\therefore$2w +14 = 7w + 21 OR 7w + 49 = 2w + 3

$\therefore$5w = -7 OR 5w = - 46

$\therefore$ w = - $\frac{7}{5}$ OR w = - $\frac{46}{5}$

$\therefore$Roots of given Quadratic equation are $w=-\frac{7}{5}$ OR $w=-\frac{46}{5}$

[6]

$x \ -\frac{1}{x}=\frac{21}{10}$

By using Vilokanam,

Here, R.H.S. = $\frac{21}{10}$ can be split as the difference of two reciprocals $\frac{5}{2}-\frac{2}{5}$

$\therefore \ x \ -\frac{1}{x}=\frac{5}{2}-\frac{2}{5}$

$\therefore$ $x=\frac{5}{2}$ OR $x=-\frac{2}{5}$

$\therefore$ Roots of given Quadratic equation arex $=\frac{5}{2}$ OR $x=-\frac{2}{5}$

[7]

$\frac{5x+7}{5x-7}+\frac{5x-7}{5x+7}=\frac{24}{35}$

By using Vilokanam,

Here, the R.H.S. $=\frac{24}{35}$ can be split as the difference of two reciprocals $\frac{7}{5}-\frac{5}{7}$

$$\therefore \frac{5x+7}{5x-7}+\frac{5x-7}{5x+7}=\frac{24}{35}$$

$$\therefore \frac{5x+7}{5x-7}=\frac{7}{5} \qquad \text{OR} \qquad \frac{5x+7}{5x-7}=-\frac{5}{7}$$

∴ 5(5x +7) = 7(5x-7) OR 7(5x+7) = -5(5x-7)

∴∴25x + 35 = 35x - 49 OR 35x + 49 = -25x + 35

∴ 10x = 84 OR 60x = -14

$\therefore\ x=\frac{84}{10}=\frac{42}{5}$ OR $x=-\frac{14}{60}=-\frac{7}{30}$

∴ Roots of given Quadratic equation are $x=\frac{42}{5}$ OR $x=-\frac{7}{30}$

3.1.2 Second Special Type under Śūnyam□ Sāmyasamuccaye Sūtra:

To solve the quadratic equation of the type

$$\frac{kx+l}{mx+n}=\frac{px+q}{rx+s}$$

Where (k + p) = (m + r) & (l + q) = (n + s);

We can use Śūnyam□ Sāmyasamuccaye Sūtra as follows:

Consider the oneness of the sum & equating it to 0, which gives one root, of the given quadratic.

i.e. (k x + l) + (p x + q) = (m x + n) + (r x + s) = 0

Which gives one root of the given quadratic equation.

Consider the oneness of the difference and assuming that difference = 0; gives another root, of the given quadratic equation

i.e. (k x + l) - (m x + n) = (p x + q) – (r x + s) = 0;

Which gives the second root of the given quadratic equation.

Solve the following:

[8]

$$\frac{5z+2}{2z+4}=\frac{3z+5}{6z+3}$$

By using Śūnyam□ Sāmyasamuccaye Sūtra,

Consider the oneness of the sum & creating that sum = 0, which gives one root, of the given quadratic.

Here, Sum = (5z + 2) + (3z + 5) = (2z + 4) + (6z + 3) = 8z + 7

$\therefore$ 8z + 7 = 0 $\therefore$ $z=-\frac{7}{8}$

Consider the oneness of difference of two sides. Now, equate that difference = 0, which gives another root, of the given quadratic equation.

i.e. (5z + 2) – (2z + 4) = (6z + 3) – (3z + 5) = 0

$\therefore$3z - 2 = 0 $\therefore$ $z=\frac{2}{3}$

$\therefore$Roots of given Quadratic equation are $z=-\frac{7}{8}$ OR $z=\frac{2}{3}$.

[9]

$$\frac{11y+2}{13y+5}=\frac{9y+7}{7y+4}$$

By using, Śūnyam□ Sāmyasamuccaye Sūtra,

Consider the oneness of the sum, also equating that sum = 0 gives one root of the given quadratic.

i.e. (11y + 2) + (9y + 7) = (13y + 5) + (7y + 4) = 0

20y + 9 = 0

$\therefore y=-\frac{9}{20}$

Consider the oneness of the difference of both sides, by equating that difference to 0 gives another root, of the given quadratic equation.

i.e. (13y + 5) – (11y + 2) = (7y + 4) – (9y + 7) = 0

2y + 3 = 0

$\therefore y=-\frac{3}{2}$

$\therefore$ Roots of given Quadratic equation are $y=-\frac{9}{20}$ OR $y=-\frac{3}{2}$

[10]

$$\frac{3x-2}{7x-5}=\frac{9x-6}{5x-3}$$

By using, Śūnyam□ Sāmyasamuccaye Sūtra,

Consider the oneness of the sum to find one of the roots of the given quadratic equation.

i.e. $(3x - 2) + (9x - 6) = (7x - 5) + (5x - 3) = 0$

$\therefore 12x - 8 = 0$

$\therefore x=\frac{8}{12}=\frac{2}{3}$

Consider the oneness of both sides by taking the difference, which gives another root, of the given quadratic equation.

i.e. $(7x - 5) - (3x - 2) = (9x - 6) - (5x-3) = 0$

$\therefore 4x - 3 = 0 \therefore x=\frac{3}{4}$

$\therefore$Roots of given Quadratic equation are $x=\frac{2}{3}$ OR $x=\frac{3}{4}$.

3.1.3 Third Special Type under Śūnyamanyat Sūtra & Śūnyam□ Sāmyasamuccaye Sūtra:

The quadratic equation of the type

$$\frac{k}{ex+a}+\frac{l}{fx+b}=\frac{r}{gx+c}+\frac{s}{hx+d}$$

Here, all letters other than x are constants.

If e = f = g = h = 1and if in the denominator the co-efficient of x of all the terms are not equal.

I.e. if e, f, g, h are not equal then by taking L.C.M. we can make it equal.

If the following conditions are satisfied.

$\frac{k}{e}+\frac{l}{f}=\frac{r}{g}+\frac{s}{h}$ & $\frac{k}{a}+\frac{l}{b}=\frac{r}{c}+\frac{s}{d}$

Then According to Śūnyamanyat Sūtra zero is one root of the given quadratic equation; the other elements are same and Add denominator on respective sides.

If its addition is equal, then according to Śūnyam□ Sāmyasamuccaye Sūtra,

That addition = 0 is another root of the given quadratic equation.

In this case, another root is,

$(gx + c) + (hx + d) = (ex + a) (fx + b) = 0$

[11]

$$\frac{3}{w+3}+\frac{4}{w+4}=\frac{5}{w+5}+\frac{2}{w+2}$$

The algebraic proof of the above equation is as follows:

By using simple division,

$\frac{3}{w+3}=1-\frac{w}{w+3}$ & so on.

$$\therefore\left(1-\frac{w}{w+3}\right)+\left(1-\frac{w}{w+4}\right)=\left(1-\frac{w}{w+5}\right)+\left(1-\frac{w}{w+2}\right)$$

∴Removing 1-1 and 1-1 from both sides, we get

$$\therefore\left(\frac{w}{w+3}\right)+\left(\frac{w}{w+4}\right)=\left(\frac{w}{w+5}\right)+\left(\frac{w}{w+2}\right)$$

∴The common factor of all terms, w = 0 and on its removal

$$\therefore\frac{1}{w+3}+\frac{1}{w+4}=\frac{1}{w+5}+\frac{1}{w+2}$$

By taking L.C.M.

$$\therefore\frac{2w+7}{w^2+7w+12}-\frac{2w+7}{w^2+7w+10}=0$$

$$\therefore(2w+7)\left[\frac{1}{w^2+7w+12}-\frac{1}{w^2+7w+10}\right]$$

$$\therefore(2w+7)\left[\frac{(w^2+7w+10)-(w^2+7w+12)}{(w^2+7w+12)(w^2+7w+10)}\right]=0$$

$$\therefore(2w+7)\left[\frac{w^2+7w+10-w^2-7w-12}{(w^2+7w+12)(w^2+7w+10)}\right]=0$$

$$\therefore(2w+7)\left[\frac{-2}{(w^2+7w+12)(w^2+7w+10)}\right]=0$$

$-2\,(2w+7)=0$

$2w+7=0$

$$\therefore w=-\frac{7}{2}$$

$$\therefore w=0\ \text{ OR } w=-\frac{7}{2}$$

These values of w are solutions of given equation.

OR

By using Vedic Sūtra:

$$\frac{3}{w+3}+\frac{4}{w+4}=\frac{5}{w+5}+\frac{2}{w+2}$$

Here, both the conditions are satisfied.

$$\frac{3}{1}+\frac{4}{1}=\frac{5}{1}+\frac{2}{1}\ \&\ \frac{3}{3}+\frac{4}{4}=\frac{5}{5}+\frac{2}{2}$$

∴ According to the Sunyamanyat Sūtra,

One root of given equation is w = 0.

Also, (w + 3) + (w + 4) = (w + 5) + (w + 2) = 2w + 7

∴ As per Śūnyam Sāmyasamuccaye Sūtra,

That addition = 2w + 7 = 0

∴ The another root is $w=-\frac{7}{2}$

∴ The solution w = 0 OR $w=-\frac{7}{2}$.

[12]

$$\frac{1}{2x+1}+\frac{1}{5x+1}=\frac{2}{5x+2}+\frac{3}{10x+3}$$

Algebraic proof is as follows:

By using simple division,

$$\frac{1}{2x+1}=1-\frac{2x}{2x+1}$$

$$\therefore\left(1-\frac{2x}{2x+1}\right)+\left(1-\frac{5x}{5x+1}\right)=\left(1-\frac{5x}{5x+2}\right)+\left(1-\frac{10x}{10x+3}\right)$$

∴Removing 1-1 and 1-1 from both sides, we get

$$\therefore\left(\frac{2x}{2x+1}\right)+\left(\frac{5x}{5x+1}\right)=\left(\frac{5x}{5x+2}\right)+\left(\frac{10x}{10x+3}\right)$$

$$\therefore\left(\frac{10x}{10x+5}\right)+\left(\frac{10x}{10x+2}\right)=\left(\frac{10x}{10x+4}\right)+\left(\frac{10x}{10x+3}\right)$$

∴The common factor of all terms, 10x = 0

x = 0 and on its removal

We have

$$\therefore \left(\frac{1}{10x+5}\right)+\left(\frac{1}{10x+2}\right)=\left(\frac{1}{10x+4}\right)+\left(\frac{1}{10x+3}\right)$$

$$\therefore \left(\frac{20x+7}{(10x+5)(10x+2)}\right)=\left(\frac{20x+7}{(10x+4)(10x+3)}\right)$$

$$\therefore \left(\frac{20x+7}{(10x+5)(10x+2)}\right)-\left(\frac{20x+7}{(10x+4)(10x+3)}\right)=0$$

$$\therefore (20x+7)\left[\frac{(100x^2+70x+12)-(100x^2+70x+10)}{(100x^2+70x+10)(100x^2+70x+12)}\right]=0$$

$$\therefore (20x+7)\left[\frac{2}{(100x^2+70x+10)(100x^2+70x+12)}\right]=0$$

$$\therefore 2(20x+7)=0$$

$$\therefore (20x+7)=0$$

∴ Another root is $x=-\frac{7}{20}$

∴ The two roots of given quadratic are $x=0$ OR $x=-\frac{7}{20}$

By using Vedic Sūtra:

$$\frac{1}{2x+1}+\frac{1}{5x+1}=\frac{2}{5x+2}+\frac{3}{10x+3}$$

Here,

$$\frac{1}{2}+\frac{1}{5}=\frac{2}{5}+\frac{3}{10}\therefore\frac{7}{10}=\frac{7}{10}\ \&\ \text{also}\ \frac{1}{1}+\frac{1}{1}=\frac{2}{2}+\frac{3}{3}$$

both the conditions are satisfied**.**

Therefore, according to the Śunyamanyat Sūtra one root is zero; and

We can write left side as below:

$$\frac{1}{2x+1}=1-\frac{2x}{2x+1};\ \frac{1}{5x+1}=1-\frac{5x}{5x+1};$$

And also right side can be written as below:

$$\frac{2}{5x+2}=1-\frac{5x}{5x+2};\ \frac{3}{10x+3}=1-\frac{10x}{10x+3}$$

Now comparing both the sides after modification,

$$\frac{2x}{2x+1}=\frac{5x}{5x+1}=\frac{5x}{5x+2}=\frac{10x}{10x+3}$$

Taking LCM,

$$\frac{10x}{10x+5}=\frac{10x}{10x+2}=\frac{10x}{10x+4}=\frac{10x}{10x+3}$$

$(10x + 5) + (10x + 2) = (10x + 4) + (10x + 3) = 20x + 7$

According to the Śūnyam□ Sāmyasamuccaye Sūtra,

$20x + 7 = 0.$

∴ The another root of the given equation is $x = -\frac{7}{20}$

∴ The two roots of quadratic are $x = 0$ **OR** $x = -\frac{7}{20}$

[13]

$$\frac{a-c}{x+a-c}+\frac{c-b}{x+c-b}=\frac{a+c}{x+a+c}+\frac{-c-b}{x-c-b}$$

Here,

$$\frac{a - c}{1}+\frac{c - b}{1}=\frac{a + c}{1}+\frac{- c - b}{1}$$

And also,

$$\frac{(a - c)}{(a - c)}+\frac{(c - b)}{(c - b)} = \frac{(a + c)}{(a + c)}+\frac{(- c - b)}{(- c - b)}$$

Therefore, according to the Sunyamanyat Sūtra one root does not have a nonzero value;

According the Śūnyam□ Sāmyasamuccaye Sūtra,

$2x + a - b = 0$

∴ Another root is $x = -\frac{1}{2}(a - b)$

∴Two roots of given quadratics are,

$$x = 0 \text{ OR } x = -\frac{1}{2}(a - b)$$

3.1.4 Fourth Special Type under Śūnyamanyat & Parāvartya Sūtra:

The equation containing all the letters except x having fixed value as below:

$$\frac{h}{ex+a}+\frac{i}{fx+b}=\frac{j}{gx+c}$$

One root of the given quadratic equation is x = 0; if

$$\frac{h}{a}+\frac{i}{b}=\frac{j}{c}\ ;$$

The quadratic equation in which two terms N_1 & N_2 on L.H.S. is compared with the right side term N_3 i.e. $N_1 + N_2 = N_3$

Then Parāvartya Sūtra is applied to combine both the sides.

If $N_1 + N_2 \neq N_3$, then co-efficient of x in all the lower terms must be different, we can make it equal.Also check

$$\frac{h}{e}+\frac{i}{f}=\frac{j}{g}$$

If it is in the above form, then Parāvartya Sūtra is applied to combine both the sides.

The right side constant term is subtracted from the left side constant term & multiply those remainders by the corresponding numerators of the term on the Left counterpart.

After this process of assimilation equated with 0, which can be written in the form

$$\frac{h(a-c)}{ex+a}+\frac{i(b-c)}{fx+b}=0$$

After applying merger Sūtra, if both the numerators are equal.

I.e. if the numerator h (a - c) = i (b - c) then they can be removed, and by applying the Sunyamanyat Sūtra $D_1 + D_2 = 0$ is another root of the given quadratic equation.

After applying the merger Sūtra, if both the numerators are not equal but different i.e. h (a - c) ≠ i (b - c), then Śūnyamanyat Sūtra will not apply.

Check co-efficient of x.

If e = f = 1, then one root of given equation is

$$x=\frac{-hb(a-c)-ia(b-c)}{h(a-c)+i(b-c)}$$

If e = f ≠ 1, i.e. other than 1 then root of equation is

$$x=\frac{-hb(a-c)-ia(b-c)}{e[h(a-c)+i(b-c)]}$$

Solve the following:

[14]

$$\frac{7}{z+7}+\frac{3}{z+3}=\frac{10}{z+10}$$

The algebraic proof of the above equation is as follows:

$$\therefore \frac{7(z+3)+3(z+7)}{(z+7)(z+3)}=\frac{10}{z+10}$$

Simplifying

$$\therefore \frac{(7z+21+3z+21)}{(z^2+10z+21)}=\frac{10}{z+10}$$

$$\therefore \frac{10z+42}{(z^2+10z+21)}=\frac{10}{z+10}$$

$10z^2 + 100z + 42z + 420 = 10z^2 + 100z + 210$

$42z = - 210$

$$\therefore z=-\frac{210}{42}=-5 \therefore z=-5$$

By Vedic Sūtra:

Here, $\frac{7}{7}+\frac{3}{3}\neq\frac{10}{10}$

Therefore, z = 0 is not a root of the given quadratic equation.

$N_1 + N_2 = 7 + 3 = 10 = N_3$

$\therefore D_1 + D_2 = 0$

i.e. $(z + 7) + (z + 3) = 2z + 10 = 0$ is root of the given quadratic equation.

$\therefore$ z = -5 is root of given equation.

By merger method,

$$\therefore \frac{7(-3)}{z+7}+\frac{3(-7)}{z+3}=0$$

$$\therefore \frac{(-21)}{z+7}+\frac{(-21)}{z+3}=0$$

$$\therefore z=\frac{-(-21)(3)-(-21)(7)}{(-21)+(-21)}=\frac{21(3+7)}{-21(1+1)}=\frac{10}{-2}=-5$$

$\therefore z=-5$ is root of given equation.

[15]

$$\frac{9}{y+3}+\frac{16}{y+4}=\frac{49}{y+7}$$

$$\frac{9}{3}+\frac{16}{4}= \frac{49}{7} \therefore 3+4=7$$

By division,

$$3-\frac{3}{\mathrm{y}+3}+4-\frac{4}{\mathrm{y}+4}=7-\frac{7}{\mathrm{y}+7}$$

∴ y = 0.

This can be verified by mere observation.

$$\frac{3}{\mathrm{y}+3}+\frac{4}{\mathrm{y}+4}=\frac{7}{\mathrm{y}+7}$$

By merger method,

$$\frac{3(-4)}{\mathrm{y}+3}+\frac{4(-3)}{\mathrm{y}+4}=0$$

$$\therefore \frac{(-12)}{\mathrm{y}+3}+\frac{(-12)}{\mathrm{y}+4}=0$$

$$\therefore \mathrm{y}=\frac{-(12)(4)-(12)(3)}{-12-12}=\frac{-12(4+3)}{-12(1+1)}=\frac{7}{2}=3\frac{1}{2}$$

∴ The two roots are $\mathrm{y}=0$ OR $\mathrm{y}=3\frac{1}{2}$.

[16]

$$\frac{6}{3\mathrm{w}+2}+\frac{3}{2\mathrm{w}+3}=\frac{4}{6\mathrm{w}+1}$$

L.C.M. = 6

$$\frac{12}{6\mathrm{w}+4}+\frac{9}{6\mathrm{w}+9}=\frac{4}{6\mathrm{w}+1}$$

$$\because \frac{12}{4}+\frac{9}{9}=\frac{4}{1}$$

∴ One solution of above quadratic is w = 0.

Applying merger method,

$$\therefore \frac{12(3)}{6\mathrm{w}+4}+\frac{9(8)}{6\mathrm{w}+9}=0$$

$$\therefore \frac{36}{6\mathrm{w}+4}+\frac{72}{6\mathrm{w}+9}=0$$

$$\therefore \frac{1}{6\mathrm{w}+4}+\frac{2}{6\mathrm{w}+9}=0$$

$\therefore 6\text{w} = \dfrac{-(1)(9)-(2)(4)}{1+2}$

$\therefore 6\text{w} = -\frac{17}{3}$

$\therefore \text{w} = -\dfrac{17}{18}$

$\therefore \text{w} = 0$ OR $\text{w} = -\frac{17}{18}$ are roots of given quadratic equation.

3.2 Factorization:

Quadratic expression $Ax^2 + Bx + C$; which uses variable x with highest degree two with always non-zero co-efficient; either B & C or both may be zero or not.

Standard formula of Quadratic equation can be obtained by equating the above expression to 0.

If an example of quadratic equation is given in the form $Ax^2 + Bx = C$ then first take all the term one side and write it in the form equal to zero and then factorize it. Factorization is one of the methods for solving quadratic equation.

3.2.1 Factorization of a Quadratic expression in one variable:

According to the current system, if we want to factorize the Quadratic expression which has prefix of x with highest degree 2 as unity. For solving quadratics in one variable $Ax^2 + Bx + C$, divide the term B in parts such that addition of that two numbers = the coefficient of variable x and the independent quantity equals the multiplication value.

[1]

Factorize: $z^2 + z - 12$

Here, co-efficient of z^2 is unity. Divide the co-efficient of z (i.e. +1) in two parts such that addition of that is +1 and the multiplication of two parts is - 12.

Here, $(+4 - 3) = +1$

& $(+4)(-3) = -12$

$z^2 + z - 12$

$= z^2 + 4z - 3z - 12$

$= z(z + 4) - 3(z + 4)$

$= (z - 3)(z + 4)$

We can calculate the product of $(z - 3)$ and $(z + 4)$ orally, which verifies to $z^2 + z - 12$.

Factors of 12 is 1, 2, 3, 4, 6, and 12.

If we select $+ 4$ and $- 3$ whose sum is $+ 1$ and product is $- 12$.

We can verify it by vertically and crosswise Sūtra

$z + 4$

$\underline{z - 3}$

$z^2 + z - 12$

Thus, **Factors of quadratic expression $z^2 + z - 12 = (z + 4)(z - 3)$**

[2]

Factorize: $2y^2 + 3y - 5$

Here, co-efficient of y^2 is 2 not unity. Evaluate two sub parts such that addition of that is $+ 3$ and multiplication of that is $- 10$.

$(- 2 + 5) = + 3$ and $(- 2)(5) = - 10$

$2y^2 + 3y - 5 = (2y^2 - 2y + 5y - 5)$

$= 2y(y - 1) + 5(y - 1)$

$= (y - 1)(2y + 5)$

In order to find the sub parts of an equation with the highest degree prefix other than 1, mental process is not possible. But by using Vedic Sub-Sūtra Ānurūpyeṇa (means ‘Proportionately’) first factor can be found and by using Ādyamadyenāntyamantyena (means ‘first by the first and last by last’) second factor can be found easily.

Steps for factorization by using Vedic Sūtras:

Divide the central term in two sub parts P_1 and P_2 so that

Ratio $= A/P_1 = P_2/C$………….. [A]

Given that $B = P_1 + P_2$

$C = P_1.P_2$

We need to find the two roots t and s for the equation. Thus,

$(x - t)(x - s) = Ax^2 + Bx + C$

The ratio [A] is the 1st factor (x – t) of given quadratic expression.

(x – t) is one factor of quadratic expression $Ax^2 + Bx + C$ found by Vedic Sub-Sūtra Ānurūpyena. The second factor is obtained by Vedic Sub-Sūtra ‘first by the first and last by last’.

2nd Factor = $(x – s) = Ax^2/x + C/ (- t)$….. [B]

In the above expression $2x^2 + 3x - 5$, the middle term + 3 is divided in two parts - 2 &
5 so that

$$\frac{2}{-2} = \frac{5}{-5} = \frac{1}{-1}$$

The ratio is 1: -1

I.e. x - 1 is one factor found by using Vedic Sub-Sūtra Ānurūpyeṇa.

The second factor is obtained by Vedic Sub-Sūtra ‘first by the first and last by last’ along with 1st factor (x – 1). Applying [B]

$$\frac{2x^2}{x} - \frac{5}{-1} = 2x + 5$$

The second factor is (2x + 5).

Thus, Factors of quadratic expression $2x^2 + 3x -5 = (x - 1) (2x + 5)$

An Alternative method:

We can also split the middle co-efficient 5 in two parts 5 & - 2 such that by using [A]

$$\frac{2}{5} = \frac{2}{5}$$

Ratio is 2: 5.

i.e. 2x + 5 is one factor found by using Vedic Sub-Sūtra Ānurūpyeṇa.

If 2x + 5 is one factor of quadratic expression $2x^2 + 3x - 5$ found by Vedic sub-Sūtra used in step [1].

The second factor is obtained by Vedic Sub-Sūtra ‘first by the first and last by last’

$$\frac{2x^2}{2x} + \frac{5}{-5} = x - 1.$$

Thus second factor is (x-1).

Thus, Factors of quadratic expression $2x^2 + 3x - 5 = (2x + 5) (x - 1)$

3.2.2 Factorization of a Quadratic expression in two variables:

The same method of factorization of one variable is used to factorize the quadratic expression in two variables x and y whose first coefficient (coefficient of x^2) is not unity.

[3]

Factorize: $7x^2 - 6xy - y^2$

Divide the central term prefix in two sub parts. Now, with the help of [A], calculate the ratio whch in turn gives the 1st factor of given quadratic expression.

In the above expression $7x^2 - 6xy - y^2$, divide the middle term -6 into 1 and -7

So that

$$\frac{7}{1} = \frac{-7}{-1} = \frac{7}{1}$$

That ratio is 7:1.

i.e. $(7x + y)$ is one factor found by using Vedic Sub-Sūtra Ānurūpyeṇa.

The second factor is obtained by using [B] as follows:

$7x + y$ is one factor of quadratic expression $7x^2 - 6xy - y^2$ founded by Vedic Sub-Sūtra as in above step. The second factor is obtained by Vedic Sub-Sūtra 'first by the first and last by last'.i.e. second factor is

$$\frac{7x^2}{7x} - \frac{y^2}{y} = x - y$$

Thus second factor is $(x - y)$.

Thus, Factors of quadratic expression $7x^2 - 6xy - y^2 = (7x + y)(x - y)$

An Alternative method:

We can also split the mid coefficient - 6 in two subparts - 7 & 1. By using [A], ratio is

$$\frac{7}{-7} = \frac{1}{-1} = \frac{1}{-1}$$

That is ratio = 1:-1

$(x - y)$ is one factor found by using Vedic Sub-Sūtra Ānurūpyeṇa means 'Proportionately'.

If (x – y) is one factor of quadratic expression$7x^2 - 6xy - y^2$ found by Vedic Sub-Sūtra Ānurūpyeṇa.

The second factor is obtained by Vedic Sub-Sūtra ‘first by the first and last by last’ and the 1st factor.

$$\frac{7x^2}{x} - \frac{y^2}{-y} = 7x + y$$

Thus, second factor is (7x + y).

Thus, Factors of quadratic expression $7x^2 - 6xy - y^2 = (x - y)(7x + y)$

[4]

Factorize: $12x^2 - 23xy + 10y^2$

Divide the middle term - 23 into - 15 and - 8 such that ratio of the first coefficient to that first part 12: 15 i.e. 4: 5 is same as the ratio of that second part to the last co-efficient 8 : 10 = 4 : 5. That ratio 4: 5 i.e. (4x - 5y) is one factor found by using Vedic Sub-Sūtra Ānurūpyeṇa.

The 2nd factor is obtained as follows:

(4x - 5y) is one factor of quadratic expression $12x^2 - 23xy + 10y^2$ found by Vedic Sub-Sūtra as above step. The 2nd factor is obtained by Vedic Sub-Sūtra Ādyam,

$$\frac{12x^2}{4x} + \frac{10y^2}{-5y} = 3x - 2y$$

The 2nd factor is (3x-2y).

Thus, Factors of quadratic expression $12x^2 - 23xy + 10y^2 = (4x - 5y)(3x - 2y)$

[5]

Factorize: $15x^2 - 14xy - 8y^2$

Divide the middle term - 14 into - 20 and 6 so that ratio is

$$\frac{15}{-20} = \frac{6}{-8} = \frac{3}{-4}$$

That is ratio is 3: -4 i.e. (3x - 4y) is one factor found by using Vedic Sub-Sūtra ‘Proportionately’.

The second factor is obtained as follows:

3x - 4y is one factor of quadratic expression $15x^2 - 14xy - 8y^2$ found by Vedic Sub-Sūtra as above step. The second factor is obtained by Vedic Sub-Sūtra ‘first by the first & last by last’, 2nd Factor is

$$\frac{15x^2}{3x} - \frac{8y^2}{-4y} = 5x + 2y$$

The second factor is (5x + 2y).

Thus, Factors of quadratic expression $15x^2 - 14xy - 8y^2 = (3x - 4y)(5x + 2y)$

Guṇitasamuccayaḥ Samuccayaguṇitaḥ Sub-Sūtra:

Meaning:

According to this, the multiplication of the variable prefix addition of the expression and the summation of the preceeding constants in the final solution both are equal.

To confirm the obtained answer of the problem in multiplications, divisions and factorization of quadratic equations an additional Sub-Sūtra is Guṇitasamuccayaḥ Samuccayaguṇitaḥ.

An example of quadratic equation for verifification is given below:

$x^2 + x - 12 = (x + 4)(x - 3)$ **………………** (i)

In equation (i) add the all the co-efficients of right side

$$= 1 + 1 - 12 = -10 \text{ and}$$

The multiplication of the addition of the co-efficient in the left side

$$= (1 + 4)(1 - 3) = 5(-2) = -10$$

$7x^2 - 6xy - y^2 = (7x + y)(x - y)$ …………….. (ii)

In equation (ii) sum of the coefficient in the factor = 7 – 6 - 1= 0 and

The multiplication of the summation = (7 + 1) (1-1) = 8(0) = 0

By using the above Sūtra both the sides of all the equations are equal

Thus, above quadratic equations are verified.

3.2.3 Solution of multivariabe Quadratics:

Equations in homogeneous form having greater than 2 variables lead to tedious & lengthy computations.

Ādyamadyenāntyamantyena & Lopanasthapanābhyām can be applied for ease of calculation of roots of given equation.

Procedure of factorization of long homogeneous quadratic expression with three variables x, y & z:

[1] Initially, discarding of z is done by substituting z as zero, so we obtained the quadratic with remaining two variables which can be solved by using Ādyamadyenāntyamantyena;
[2] Similarly equating y to zero eliminates it. Thus, we obtaine the quadratic with variables other than y which can be factorized by using the above sub-Sūtra used in step [1];
[3] By using the results obtained in [1] and [2], the final solution is arrived at by identifying the pattern and combining the common terms.

To factorize long homogeneous quadratic expression with three variables x, y & z, only two elimination by eliminating one letter at a time by three possible ways.
[a]Out of given 3 variables by consecutive exclusion of x and then y we get two expressions whose solutions contains two variables y & z and x & z respectively. After merging & filling the pattern, the final terms are obtained successfully.
[b] Repeat these steps of [a] after eliminating y and then z.
[c] Discard first x and then z.Reiterate the above process [a].

By taking examples procedure of factorization long quadratic expression can be explained:

[6]
Factorize: $2u^2 + v^2 - 3w^2 - 3uv + 2vw - wu$

Let Q = $2u^2 + v^2 - 3w^2 - 3uv + 2vw - wu$
First by considering ($w = 0$); we obtain the expression in terms of u and v.
Factorize that expression with the help of adyam Sūtra, we get
If $w = 0$, then $[Q] = 2u^2 - 3uv + v^2$
$= (u - v)(2u - v)$

Similarly, eliminate v $(v = 0)$; factorizing the obtained expression in terms of u and w.

If $v = 0$, then $[Q] = 2u^2 – wu - 3w^2$

$= (u + w)(2u - 3w)$

Connecting the like terms caused by eliminating w and v, we get

$Q = (u - v + w)(2u - v - 3w)$

OR

If $u = 0$, then $[Q] = v^2 + 2vw - 3w^2$

$= (v - w)(v + 3w)$

Similarly, if $v = 0$, then as as above $[Q] = 2u^2 - wu - 3w^2 = (u + w)(2u - 3w)$

With the help of two set of factors obtained after eliminating u & v; by filling the gaps among that two factors we can get

$$Q = (u - v + w)(2u - v - 3w)$$

OR

Substituting $w = 0$ in [Q],

$2u^2 - 3uv + v^2$

$= (u - v)(2u - v)$

Similarly, if $u = 0$, then [Q] is $v^2 + 2vw - 3w^2$

$= (-v - 3w)(w - v)$

By filling the breaks of factors which are obtained after eliminating w and u

$$\mathbf{Q = (u - v + w)(2u - v - 3w)}$$

[7]

Factorize: $2x^2 + y^2 - 6z^2 + 2xy - 10yz + 4zx$

Let $R = 2x^2 + y^2 - 6z^2 + 2xy - 10yz + 4zx$

If $z = 0$, $[R] = 2x^2 + 2xy - 4y^2$

$= (x + 2y)(2x - 2y) = 2(x + 2y)(x – y)$

Similarly,

If $y = 0$, then $[R] = 2x^2 + 4zx - 6z^2$

$= (x - z)(2x + 6z) = 2(x - z)(x + 3z)$

i.e. $[R] = 2(x + 2y)(x - y) = 2(x - z)(x + 3z)$

By filling the gaps among the factors after elimination of z & y

$[R] = 2(x - y + 3z)(x + 2y - z)$ OR $[R] = 2(x - y - z)(x + 2y + 3z)$

Also by eliminating x, we get factors in terms of y and z;

If $x = 0$, then $[R] = - 4y^2 -10yz - 6z^2$

$= (y + z)(- 4y - 6z) = 2(y + z)(- 2y - 3z)$

& If $y = 0$, then as above $[R] = (x - z)(2x + 6z) = 2(z - x)(-x -3z)$

i.e. $[R] = 2(y + z)(-2y - 3z) = 2(z - x)(- x - 3z)$ after discarding x and y

$\therefore$ $[R] = 2(-x + y + z)(-x - 2y - 3z)$

Thus, the final solution is $R = 2(x - y - z)(x + 2y + 3z)$

[8]

Factorize: $z^2 - 2x^2 - 3y^2 - 5xy + 2yz + zx$

Let $S = z^2 - 2x^2 - 3y^2 - 5xy + 2yz + zx$

If $x = 0$, then $[S] = z^2 - 2yz - 3y^2$

$= (z - y)(z + 3y)$

Similarly,

If $y = 0$, then given expression $= z^2 + zx - 2x^2$

$= (z - x)(z + 2x)$

$[S] = (z + 3y)(z - y)$ OR $[S] = (z + 2x)(z - x)$

By filling up the gaps between the factors obtained after elimination of x and y.

$[S] = (z - x - y)(z + 2x + 3y)$ OR $[S] = (z - x + 3y)(z - y + 2x)$

Also, eliminating z we get factors in x and y, i.e. $[S] = -2x^2 - 5xy - 3y^2$

$= (x + y)(-2x - 3y) = (-x - y)(2x + 3y)$

Also, as above $y = 0$, then $[S] = (z - x)(z + 2x)$

i.e. $[S] = (-x - y)(2x + 3y) = (z - x)(z + 2x)$

After filling the break between two factors obtained because of the elimination of two variables z & y.

Thus, the final solution is $S = (z - x - y)(z + 2x + 3y)$

Procedure of factorization of long homogeneous quadratic expression with three variables u, v, w & Independent term:

Consider 3 variables (u, v, w) along with an independent term for the process of factorization.Out of the 4 terms there arise 3 possible combinations for exclusion of 2 terms satisfying the condition of retention of the pair containing an independent term & the remaining variable.

First exclude u and v by substituting the values of both u and v as zero, thereby getting a retention pair of an independent term and w. Similarly, excluding u and w, we get quadratic in terms of v & an independent term.

In this way, the third pair can be obtained by eliminating v and w.Factorizing that quadratics by Sūtra Adyam 3 different factors are obtained. By filling the gaps between the obtained three factors the final factors of the given expression in terms of 3 variables along with independent term can be found.

[9]

Factorize: $2u^2 - 3v^2 - 4w^2 - uv - 7vw - 2wu - 5u - 10v - 11w - 7$

Let $E = 2u^2 - 3v^2 - 4w^2 - uv - 7vw - 2wu - 5u - 10v - 11w - 7$

Here, there are three letters u, v, w and independent term, and 3 elimination by eliminating two letters at a time so retain 1 remaining letter & independent term each time.

First eliminate u, v by putting $u = v = 0$ and retain only in w and independent term.

After factorizing it given expression

$E = -4w^2 - 11w - 7 = (-4w - 7)(w + 1)$

Then, by eliminating v & w, the quadratic is obtained in variable u along with the independent term & then factorize it.

$\therefore$ Given expression $E = 2u^2 - 5u - 7 = (2u - 7)(u + 1)$

Then, by eliminating u & w, we can obtained the quadratic in variable v along with the independent term & then factorize it.

$\therefore$ Given expression $E = -3v^2 - 10v - 7 = (-3v - 7)(v + 1)$

With these three sets of factors,

$\therefore$ Given expression $E = (-4w - 7)(w + 1) = (2u - 7)(u + 1) = (-3v - 7)(v + 1)$

By filling the breaks among the above 3 sets of factors, the final factors can be found with 3 variables along with independent tem.

The final solution is $E = (u + v + w + 1)(2u - 3v - 4w - 7)$

3.2.4 Factorization of Quadratics expression with four variables:

Solving quadratic equations containing more than three variables is quite a challenging task especially for the equations in homogeneous form.

Let t, u, v and w be four variables.

Vedic Ādyamadyenāntyamantyena and Lopanasthāpanābhyām Sub-Sūtra can be utilized in order to solve intricate problems in a relatively easier manner.

Procedure of factorization of long homogeneous quadratic expression with four variables x, y, z & w:

To factorize long homogeneous quadratic expression with four variables x, y, z and w, three times by eliminating two letters at a time.

Either eliminate x, y by putting $x = y = 0$, then we will get the expression in other two remaining variables z & w.

Eliminate y, w by putting $y = w = 0$ retain only in x, z or eliminate x, w by putting

$x = w = 0$ and retain only in y, z.

First eliminate x, y and retain only in z, w; eliminate y, z by retain only in x, w; or by eliminating x & z to retain it in y & w.

First eliminate x, w; then eliminate z, w, and then eliminate x, z.

First eliminate y, w; then eliminate z, w and then eliminate y, z.

After eliminating two letters at a time, the quadratic with remaining two letters are obtained. By factorizing and merging it along with filling the gaps the final factors in all 4 letters can be found.

[10]

Factorize: $2x^2 + 7y^2 + 3z^2 - 5w^2 - 9xy - 10yz - 5xz - 3wx + 2wz - 2yw$

Let $T = 2x^2 + 7y^2 + 3z^2 - 5w^2 - 9xy - 10yz - 5xz - 3wx + 2wz - 2yw$

When there are four letters x, y, z and w three elimination by eliminating two letters at a time. There are four possible ways to factorize.

[I]

First we eliminate x, y by putting $x = y = 0$ and retain only in z, w;

$[T] = 3z^2 + 2wz - 5w^2$

$\quad = (z - w)(3z + 5w)$

Then eliminate y, z by putting $y = z = 0$ retain only in x, w;

$[T] = 2x^2 - 3wx - 5w^2$

$\quad = (x + w)(2x - 5w) = (-x - w)(-2x + 5w)$

Also, if $x = z = 0$, then

$[T] = 7y^2 - 2yw - 5w^2$

$\quad = (y - w)(7y + 5w)$

$T = (z - w)(3z + 5w) = (- x - w)(-2x + 5w) = (y - w)(7y + 5w)$

Filling the gapes in,

The solution is $(z - x + y - w)(3z - 2x + 7y + 5w)$

[II]

First eliminate x, w by putting $x = w = 0$ and retain only in y, z;

$[T] = 7y^2 + 10yz + 3z^2$

$= (y + z)(7y + 3z)$

By eliminating z & w (i.e. $z = w = 0$)

$[T] = 2x^2 - 9xy + 7y^2$

$= (x - y)(2x - 7y) = (y - x)(7y - 2x)$

If $x = z = 0$, then

$[T] = 7y^2 - 2yw - 5w^2$

$= (y - w)(7y + 5w)$

With these three sets of factors,

Thus, $T = (y + z)(7y + 3z) = (y - x)(7y - 2x) = (y - w)(7y + 5w)$

Filling in the gaps,

The solution is $T = (y + z -x - w)(7y + 3z - 2x + 5w)$

[III]

By eliminating z & w ($z = w = 0$) as above

$T = (y - x)(7y - 2x)$

By eliminating y & z (i. e. substituting $y = z = 0$),

$T = (- x - w)(-2x + 5w)$

Then, eliminate y, w by putting $y = w = 0$ and retain only in x, z;

If $y = w = 0$, then given expression $T = 2x^2 - 5xz + 3z^2$

$= (x - z)(2x - 3z)$

$= (z - x)(3z - 2x)$

With these three sets of factors,

$T = (y - x)(7y - 2x) = (- x - w)(-2x + 5w) = (z - x)(3z - 2x)$

Filling in the gaps,

The solution is $T = (y - w + z - x)(7y + 5w + 3z - 2x)$

[IV]

First eliminate x, w by putting $x = w = 0$ and retain only in y, z;

$[T] = 7y^2 + 10yz + 3z^2$

$= (y + z)(7y + 3z)$

Also, eliminate y, w by putting $y = w = 0$ and retain only in x, z;

$[T] = 2x^2 -5xz + 3z^2$

$= (x - z)(2x -3z) = (z - x)(3z - 2x)$

Then eliminate x, y by putting $x = y = 0$ and retain only in z, w;

$[T] = 3z^2 + 2wz - 5w^2$

$= (z - w)(3z + 5w)$

With these three sets of factors,

Given expression $T = (y + z)(7y + 3z) = (z - w)(3z + 5w) = (z - x)(3z - 2x)$

By filling the breaking variables, we found the final factor,

The solution is $T = (y - w - x + z)(7y + 5w - 2x + 3z)$

All four possible ways we get same answer,

$T = (-x + y + z - w)(-2x + 7y + 3z + 5w) = (x - y - z + w)(2x -7y - 3z - 5w)$

Factorization of Cubic Expressions:

[11]

Factorize: $x^3 + 2x^2 - 23x - 60$

Let $E = x^3 + 2x^2 - 23x - 60$

Sum of coefficient of all the term = 1 + 2 - 23 - 60 = - 80

Factors of 80 = 1, 2, 4, 5, 8, 10, 16, 20, 40, 80

And Factors for 60 = 1, 60, 2, 30, 3, 20, 4, 15, 5, 12, 6, 10

We want to find the factors such that product of any three is 60

& total of them must be 2.

$\therefore$ Possible factors are 3, 4,-5

Also co-efficient of x^2, 12 - 20 - 15 = - 23

$\therefore$ E = $x^3 + 2x^2 - 23x - 60 = (x + 3)(x + 4)(x - 5)$

3.3 Factorization of Cubic Equations:

We can also solve cubic equation by using Parāvartya Sūtra, Lopanasthāpanābhyām Sub-Sūtra, and Pūranapūrnabhyām Sūtra which means by completion as well as incompletion from second to fourth powers.

[12]

Solve: $z^3 + 5z^2 - z - 5 = 0$

$\therefore z^3 + 5z^2 = z + 5$

But $(z + 2)^3 = z^3 + 6z^2 + 12z + 8$

By Pūraṇapūrṇabhyām,

$\therefore$ Substituting the value of $z^3 + 6z^2$ from above, we have:

$(z + 2)^3 = z^2 + z + 5 + 12z + 8$

$= z^2 + 13z + 13$

$= z^2 + 13z + 22 - 9$

$= (z + 2)(z + 11) - 9$

$= (z + 2)(z + 2 + 9) - 9\ (z + 2)^3$

By substituting z + 2 = y, $y^3 = y(y + 9)$

$\therefore y^3 = y^2 + 9y - 9$

$\therefore y^3 - y^2 - 9y + 9 = 0$

Sum of co-efficient of all the terms = 1 - 1 - 9 + 9 = 0

$\therefore$ (y-1) is one factor of this cubic equation.

By synthetic division we can find other two factors (y + 3) and (y - 3).

Thus, y =1 and y = ± 3 but z + 2 = y

$\therefore$ z + 2 = 1 & z + 2 = ± 3

Thus, final solution is z = -1, z = 1 & z = -5

[13]

Solve: $z^3 + 12z^2 + 47z + 60 = 0$

Let $z^3 + 12z^2 = - 47z - 60$

But $(z + 4)^3 = z^3 + 12z^2 + 48z + 64$

By Pūraṇapūrṇabhyām,

Substituting the value of $z^3 + 12z^2$ from above, we have:

$(z + 4)^3 = - 47z - 60 + 48z + 64 = z + 4$

Consider z + 4 = y $\therefore$ $y^3 = y$ $\therefore$ $y^3 - y = 0$

$\therefore$ $y (y^2 - 1) = 0$ $\therefore$ $y (y -1) (y + 1) = 0$ $\therefore$ y = 0 and y = ± 1

I .e. z + 4 = 0 and z + 4 = ± 1 since z + 4 = y

$\therefore$ z = 0, z = 3 and z = -5

[14]

Solve: $2z^3- 12 z^2 + 22z - 12= 0$

$2z^3 - 12 z^2 + 22z -12 = 0$

$\therefore$ $2(z^3 - 6 z^2 + 11z - 6)$

Here, sum of all the coefficient = 1- 6 + 11 - 6 = 0

$\therefore$ z - 1 is one factor of the given expression.

Here, sum = 0 and the last term is 12 whose factors are 1, 2, 3, 4, 6, 12

We find three numbers co-efficient of $z^2 = a + b + c = - 6$

$\therefore$ We select the group -1,-2 and -3 [Since - 1- 2 - 3 = - 6]

Also testing for the co-efficient of z.

Here, (a.b + b.c + c.a) =2 + 6 + 3 = 11 = co-efficient of z.

$\therefore$ $z^3 - 6z^2 + 11z - 6 = (z - 1) (z - 2) (z - 3) = 0$

Thus, the final solution is z = 1, 2 and 3

[15]

Solve: $z^3 - 6z^2 + 5z + 12 = 0$

Here sum of co-efficient of all the term $= 1 - 6 + 5 + 12 = 12 \neq 0$

$\therefore$ $(z - 1)$ is not a factor of the given expression.

Sum of co-efficient of even power of z = Sum of co-efficient of odd power of z.

i.e. $1 + 5 = -6 + 12 = 6$

$\therefore$ $(z + 1)$ is a factor of the given expression.

$\therefore$ To find another two factors, dividing E by the factor $(z + 1)$, by using Sub - Sūtra Ādyamadyenāntyamantyena 1 as a 1st co-efficient and 12 as the last.

The co-efficient of the middle term $= 6 - (1 + 12)$

$= 6 - 13 = -7$

$\therefore$ $Q = z^2 - 7z + 12$

By using Ānurūpyeṇa and Ādyam Sub-Sūtra $Q = (z - 3)(z - 4)$

$\therefore$ $E = z^3 - 6z^2 + 5z + 12 = (z + 1)(z - 3)(z - 4) = 0$

Thus, the final solution is $z = -1, 3$ and 4

OR

Here, sum of co-efficient of all the term $= 1 - 6 + 5 + 12 = 12$ and the last term is 12

Factors of 12 = 1, 2, 3, 4, 6, 12

But we want three numbers such that their total should be - 6

$\therefore$ We select the group 1, -3, -4. [Since $1 - 3 - 4 = -6$]

Also testing for the co-efficient of z.

Here, $(a.b + b.c + c.a) = -3 + 12 - 4 = 5$ = co-efficient of z.

$\therefore$ Given expression $= z^3 - 6z^2 + 5z + 12 = (z + 1)(z - 3)(z - 4) = 0$

$\therefore$ $z = -1, 3$ and 4

[16]

Solve: $x^3 + x^2 - 19x - 34 = 0$

Here sum of co-efficient of all the term $= 1 + 1 - 19 - 34 = -51 \neq 0$

$\therefore$ $(x-1)$ is not a factor of the given expression.

Sum of co-efficient of even power of x = Sum of co-efficient of odd power of x i.e. $1 - 19 \neq 1 - 34$

$\therefore$ $(x + 1)$ is not a factor of the given expression.

By trial and error method $(x + 2)$ is one factor of the given expression.

To find another two factors, dividing E by the factor $(x + 2)$, by using

By Ādyam Sūtra, 1 is 1st coefficient & -17 is last.

$\therefore$ Co-efficient of the middle term $= -17 - (1 - 17) = -17 - (-16) = -1$

$\therefore$ $Q = x^2 - x - 17$

$\therefore$ Given expression $= x^3 + x^2 - 19x - 34 = (x + 2)(x^2 - x - 17) = 0$

Factors of $x^2 - x - 17$

$2x - 1 = \pm\sqrt{1 - 4(1)(-17)}$

$\therefore 2x - 1 = \pm\sqrt{69}$

$\therefore 2x = 1 \pm \sqrt{69}$

$\therefore x = \frac{1}{2} \pm \frac{\sqrt{69}}{2}$

i.e. The value of x above are two roots of $x^2 - x - 1$

$\therefore$ The final solution: $x = -2, x = \frac{1}{2} \pm \frac{\sqrt{69}}{2}$

OR

Here, sum of co-efficient of all the term $= 1 + 1 - 19 - 34 = -51$ and the last term is 34

Factors of $-34 = \pm 1, \pm 2, \pm 17$

But we want three numbers such that their total should be +1

$\therefore$ It is not possible to select the group of three numbers such that sum of three numbers is 1.

But if we substitute $x = -2$ then we get remainder $R = 0$.

$\therefore$ $(x + 2)$ is one factor of the given expression.

$\therefore$ To find another two factors, dividing E by the factor $(x + 2)$, by using sub- sūtra Ādyamadyenāntyamantyena orders substitution of 1 and -17 as the start & end prefixes.

$\therefore$ The middle coefficient is $-17 - (1 - 17) = -17 - (-16) = -1$

$\therefore$ $Q = x^2 - x - 17$

$\therefore$ Given expression $= x^3 + x^2 - 19x - 34 = (x + 2)(x^2 - x - 17) = 0$

Factors of $x^2 - x - 17$ and final answer can be obtained as above.

∴ The final solution is $x = -2$, $x = \frac{1}{2} \pm \frac{\sqrt{69}}{2}$

Special equations (resembling Cubics):

Equations which are of this special kind bring to mind cubic ones. Post simplifying we come to know that it is the simple equation of the first degree by using the formula
Śūnyam Samuccaya.

[17]

Solve: $(y - 2)^3 + (y - 8)^3 = 2(y - 5)^3$

The given equation is solved by using the current system,

Post rigorous simplifications and quantitative manipulations, finally after so many steps we get the answer $y = 5$. But Vedic formula gives answer at first sight.

By Śūnyam Sāmyasamuccaye Sūtra,

$(y - 2) + (y - 8) = 2(y - 5)$

Avoid the term which is numeric.

On the right hand side $(y - 5)$ is the remaining part excluding the third degree entity.

i.e. $(y - 5) = 0$

∴ The solution is $y = 5$.

[18]

Solve: $(z - 147)^3 + (z + 145)^3 = 2(z - 1)^3$

By Śūnyam Sāmyasamuccaye Sūtra,

$(z - 147) + (z + 145) = 2(z - 1)$

Avoiding the numeric term, right side $z - 1$ emerges.

i. e. $z - 1 = 0$

Therefore, $z = 1$ is a factor of the above given equation.

3.4 Factorization of biquadratic equations:

[19]

Solve: $z^4 + 3z^3 - 21 z^2 - 83z - 60 = 0$

Here, The sum of co-eff. of odd power of z = the sum of co-eff. of even power of z

$\therefore$ $3 - 83 = 1 - 21 - 60$ $\therefore$ $-80 = -80$

$\therefore$ $(z + 1)$ is one factor of given equation.

By using Synthetic division,

$z^4 + 3z^3 - 21z^2 - 83z - 60 = (z + 1)(z^3 + 2z^2 - 23z - 60)$

Now we factorize, $(z^3 + 2z^2 - 23z - 60)$

$\pm 1, \pm 60, \pm 2, \pm 30, \pm 3, \pm 20, \pm 4, \pm 15, \pm 5, \pm 12, \pm 6, \pm 10$ are factors of 60.

But we want three numbers such that their total should be 2.

We select the group 3, 4,-5 [since, $3 + 4 - 5 = 2$]

Also testing for the co-efficient of z.

Here, $a.b + b.c + c.a = 12 - 20 - 15 = -23$ = co-efficient of z.

$\therefore$ Possible factors of $-60 = (3)(4)(-5)$

$z^3 + 2z^2 - 23z - 60 = (z + 3)(z + 4)(z - 5)$

$\therefore$ Given expression $= z^4 + 3z^3 - 21z^2 - 83z - 60 = (z + 1)(z + 3)(z + 4)(z + 5) = 0$

Thus, the final solution is $z = -1, -3, -4$ and -5.

[20]

Solve: $z^4 - 8z^3 + 8z^2 + 32z - 48 = 0$

$z^4 - 8z^3 = -8z^2 - 32z + 48$

But, $(z - 2)^4 = z^4 - 8z^3 + 24z^2 - 32z + 16$

$= -8z^2 - 32z + 48 + 24z^2 - 32z + 16$ [Since $z^4 - 8z^3 = -8z^2 - 32z + 48$]

$= 16z^2 - 64z + 64$

$\therefore$ $(z - 2)^4 = 16(z - 2)^2$

By Pūraṇapūrṇabhyām,

Let $z - 2 = y$

$\therefore$ $y^4 - 16y^2 = 0$ $\therefore$ $y^2(y^2 - 16) = 0$

$\therefore$ $y^2 = 0$ OR $(y^2 - 16) = 0$

Since $z - 2 = y$, $(z - 2)^2 = 0$ OR $(z - 2)^2 - (4)^2 = 0$

The solution of $z^4 - 8z^3 + 8z^2 + 32z - 48 = 0$ is $z = 2, 2$ OR $z = 6, -2$

Special type "biquadratic":

One special type of biquadratic equation is the type in which the L.H.S. consists of the sum of the fourth power of two binomials whose exact middle binomial is possible and the R.H.S. is given.

The common format of biquadratic is $[(x + m) + n]^4 + [(x + m) – n]^4 = p$

This type of problem can be can be solved by Vya□□isam□□i□ Sūtra or the Lopanasthāpanābhyām Sub-Sūtra which teaches us by taking mean of two quantities

OR by dividing the central term the quartic can be converted in basic equation without odd powers, i.e. x^3 and x.

By using Vya□□isam□□i□ Sūtra, we can also solve the equation involving more complex numbers, fractions, surds, imaginary quantity and literal co-efficient.

[21]

Solve: $(x + 8)^4 + (x + 6)^4 = 3026$

By applying Vya□□isam□□i□ Sutra, $(x + 7) = a$ = the average of the two binomials.

$\therefore\ (a + 1)^4 + (a - 1)^4 = 3026$

$\therefore\ (a^4 + 4a^3 + 6a^2 + 4a + 1) + (a^4 – 4a^3 + 6a^2 - 4a + 1) = 3026$

$\therefore\ 2a^4 + 12a^2 + 2 = 3026 \quad \therefore\ 2a^4 + 12a^2 = 3024 \quad \therefore\ a^4 + 6a^2 - 1512 = 0$

$\therefore\ a^4 - 36a^2 + 42a^2 - 1512 = 0 \quad \therefore\ (a^2 - 36)(a^2 + 42) = 0$

$\therefore\ [(x + 7)^2 - 36] = 0$ OR $[(x + 7)^2 + 42] = 0$

$\therefore\ (x + 7)^2 = 36 = (6)^2$ OR $(x + 7)^2 = - 42$

$\therefore\ (x + 7) = \pm 6$

Since $(x + 7)^2 = - 42$ is not possible.

There will always be a non-negative value for self multiplication.

$\therefore\ (x + 1)(x + 13) = 0$ is the only possible factors.

Thus, the final solution is x = -1 and -13

[22]

Solve : $(x + 5)^4 + (x + 3)^4 = 1146$

According to Vya□□isam□□i□ Sutra, $(x + 4) = a$ = the average of the two binomials.

$\therefore\ (a + 1)^4 + (a - 1)^4 = 1146$

$\therefore\ 2a^4 + 12a^2 + 2 = 1146 \quad \therefore\ 2a^4 + 12a^2 = 1144$

$\therefore\ a^4 + 6a^2 - 572 = 0 \quad \therefore\ a^4 + 31a^2 - 25a^2 - 572 = 0$

$\therefore\ (a^2 - 25)(a^2 + 31) = 0 \quad \therefore\ a^2 = 25$ OR $a^2 = -31$

$a^2 = -31$ is not possible.

Value by exponentiation to an even power will never be negaive.

$\therefore\ a^2 = 25$ is the only possible factor.

$\therefore\ (x + 4)^2 - (5)^2 = 0$; since $(x + 4) = a$

$\therefore\ (x + 9)(x - 1) = 0$

Thus, the final solution is $x = 1, -9$

Special type of equations which look like "biquadratic":

- Some special equations whose appearance is similar to the **"**biquadratic " we come to know that it is simple equation of the first degree after writing it in a simplified form and can be solved by using Śūnyam□ Sāmyasamuccaye Sūtra.

The following are the particular characteristics of this type of equations:

[a]Out of given the four binomial the summation of the 1st & 2nd should be equal to the summation of the 3rd & 4th i.e. $N_1 + D_1 = N_2 + D_2$

[b] Absolute term of the binomials N_2, N_1, D_1, and D_2 must be in A.P.

[c] The subtraction of two right side binomials and three times the result achieved after subtracting two left side binomials must be equal.i.e. $D_2 - N_2 = 3(D_1 - N_1)$

[d]Thus, if in each direction, the addition of numerator and denominator matches, then that sum = 0 is the solution.

[23]

Solve the following:

$$\frac{(x+3)^3}{(x+4)^3} = \frac{x+2}{x+5}$$

By neglecting cube; $N_1 + D_1 = N_2 + D_2$

Here, $(x + 2)$, $(x + 3)$, $(x + 4)$ and $(x + 5)$ are in A.P and $(5 - 2) = 3\ (4 - 3)$

Also, $N_1 + D_1 = N_2 + D_2$

Therefore, by using Śūnyam Sāmyasamuccaye,

$2x + 10 = 0$

Thus the final solution is $x = - 5$.

[24]

Solve the following:

$$\frac{(x-4)^3}{(x-6)^3} = \frac{x-2}{x-8}$$

By neglecting cube; $N_1 + D_1 = N_2 + D_2$

$$N_1 + D_1 = N_2 + D_2 = 2x - 10$$

Using Śūnyam Sāmyasamuccaye,

$\therefore 2x - 10 = 0$

$\therefore\ x = 5$

Corollary:

Because of the A.P. relationship between the binomial factors, if cross-addition of these factors as the addition of both the sides equal to zero gives us the solution.

By cross- multiplication, we have

$(x-8)(x-4)^3 = (x-2)(x-6)^3$

And by cross-addition of these factors as the addition of both the terms = $4x - 20$.

$\therefore 4x - 20 = 0 \therefore x = 5$

- The second special kind of special equations similar to **“**biquadratic” but afterward simplifying it we come to know that it is simple equation of the first degree and solved by using “Śūnyam Sāmuccaye” Sūtra.

If it is in the form of

$$(x + k)\ (x + 1)(x + m)\ (x + n) = (x + p)\ (x + q)\ (x + r)\ (x + s)$$

It must contains the following characteristics:

The cross-addition must give same total on both sides

I.e. $k + l + m + n = p + q + r + s$

The summation of any two binomials on right side = the summation of any two binomials on left side.

After arranging all the terms in ascending order on both the sides,

$$k.l + m.n = p.q + r.s$$

By using “Śūnyam Sāmyasamuccaya” Sūtra, solving the total on both sides equal to zero is the required solution**.**

[25]

Solve:

$$\frac{(x+2)(x-2)}{(x+5)(x+9)} = \frac{(x+1)(x-1)}{(x+6)(x+8)}$$

$(x + 5)\ (x + 9)\ (x + 1)\ (x - 1) = (x + 2)\ (x - 2)\ (x + 8)\ (x + 6)$

The cross-addition must give same total on both sides.

i.e. $2 - 2 + 6 + 8 = 1 - 1 + 5 + 9$

The sum of any two binomial on the right side

$$= (x + 6) + (x + 1) = (x + 8) + (x - 1) = 2x + 7$$

The sum of any two binomial on the left side

$$= (x + 5) + (x + 2) = (x + 9) + (x - 2) = 2x + 7$$

After arranging all the terms in ascending order on both the sides,

(-2).2 + 6 .8 on the left = (-1) .1 + 5.9 on the right = 14

By using “Śūnyam□ Sāmyasamuccaye” Sūtra

The total on both sides = 4x + 14 = 0

∴ 4x = - 14

Therefore, the required solution is $x = -\frac{14}{4} = -\frac{7}{2}$

Chapter-4

Determinants and Matrices

4.1 Introduction:

Matrices and Determinants are important concept of Linear Mathematics.

Matrix:

A matrix is a rectangular grid of numbers, symbols or expressions that is arranged in a row or column format enclosed in square or curved brackets.

A matrix of order m × n is a matrix which contains m number of rows and n number of columns written as

a11a12…………a1na21a22…………a2n::…………:am1am2…………amnm×nor

a11a12…………a1na21a22…………a2n::……………:am1am2……………amnm×n

Different types of Matrices:

Rectangular Matrix:

A matrix containing m number of rows and n number of columns. A unique kind of matrix that follows the rule m ◷ n, is said to be of Rectangular type of order m × n.

Examples of Rectangular Matrices of order 3 × 4, 4 × 3, 3 × 2, and 2 × 3:

a11a12a21a22a31a323×2 a11a21a12a13a22a232×3

a11a12a13a14a21a22a23a24a31a32a33a343×4a11a21a12a22a13a23a31a32a33a41a51a42a52a43a534×3

Square Matrix:

Those kinds of numerical arrangements that strictly satisfy the constraint m = n throughout qualify to be named as Square matrices of order n.

Examples of Square Matrices of 2nd, 3rd & 4th order respectively, a11a12a21a222x2; a 11a 12a

13a 21a 22a 23a 31a 32a 333 × 3; a 11a 12a 13a 14a 21a 22a 23a 24a 31a 32a 33a34a 41a

42a 43a 444 × 4

Determinants:

The determinant is a unique number associated with each square matrix and it cannot be found in any other type of matrix. i.e. For finding determinant, number of rows and columns must be equal.

Examples of 2nd, 3rd and 4th order determinant:

By using the method for finding 2nd order determinant third and fourth order determinant can be obtained easily.

abcpqr2×3; abcpqrlmn3×3; abcdpqrsklmnefgh4×4

- The determinant has definite value.
- It can be also find out by using the Vertically & Crosswise Sūtra of Vedic Mathematics.

Difference between Matrix and Determinants:

The main difference is that Matrix is an array of numbers and Determinant is a single number.

The determinant has definite value while matrix has no definite value.

A matrix is a function, mapping the Cartesian product (e. g. real or complex number); while determinant is a certain scalar associated with the given matrix using a particular method.

4.2 Evaluation of determinant of matrices:

The necessary condition before evaluation dictates that the matrices must be of square type whereas for rectangular ones, determinant value is not defined. The determinant of a square

matrix (say Z) will be denoted by det (Z) or Z.

4.2.1 Determinant of third order matrix:

[1]

Find the determinant of 3 x 3 matrix A= 3 21-5 1 0 3-14

A=det(A)= 3 2 1-5 1 0 3-1 4=3 10-14-2-50 34+1-5 1 3-1

ॐ Det (A) = 3(4 - 0) - 2 (-20 - 0) +1(5 - 3)

= 3(4) - 2(- 20) + 1(2)

= 12 + 40 + 2

= 54

Determinant of 3^{rd} order matrix by Vertically & Crosswise Sūtra:

A third order determinant is a formula reducing $3^2 = 9$ figures to one. By using Vertically & Crosswise Sūtra determinant can be solved easily. For finding third order determinant first separate the first row (column) from the second and third row (column) by partition. Three second order determinants arise from the second and third row, out of them that elements spaced one row (column) apart contains negative sign: adjacent elements lead to a positive sign. For third order determinant out of the three second order determinants calculated after partitioning, the sign of second is negative.

Here, after partitioning first row from the second and third row calculated three second order determinants are

-5 1 3-1=2,-50 34= -20 & 10-14=4

The result of three second order determinants are noted in a fourth row.

A=det(A)= 3 2 1-5 1 0 3-1 4 + - + = 3 2 1-5 1 0 3-1

4…………………………. 2-204

The multiplication pattern of partitioned row and fourth new row is as follows:

3 2 1

2 -20 4

Figure: 4.1

With the help of V & C Sutra from the figure above,

ॐ Det (A) = 3(4) - 2(-20) + 1(2) = 54

We can find third order determinant by separating any one row from the remaining other two rows. We can also find third order determinant by separating any one column from the remaining other two columns. The multiplication pattern of partitioned column and fourth column becomes

A=det(A)= 3 2 1-5 1 0 3-1 4 = +13 -9 +2 3 21-5 10 3-14+- +

The multiplication pattern of partitioned column 3 with previous to 1st column new column is as follows:

13 1

-9 0

2 4

Figure: 4.2

From the above figure with the help of V & C Sutra,

ॐ Det (A) = + (13) (4) - (- 9) (0) + (2) (1) = 52 + 2 = 54

4.2.2 **Determinant of Fourth order matrix:**

[2]

Find the determinant of 4 × 4 matrix A=4557768822233494

A=det(A)=4557768822233494

A=4688223494-5788223394+5768223344-7768222349……… [P]

Where, 688223494=62394-82344+82249

= 6 (8 – 27) – 8 (8 – 12) + 8 (18 – 8)

= (-19) – 8 (- 4) + 8 (10)

= -114 + 32 + 80 = -2

788223394= 72394-82334+82239

= 7 (8 - 27) - 8 (8 - 9) + 8 (18 - 6)

= 7 (8 -27) - 8 (8 - 9) + 8 (18 - 6)

= -133 + 8 + 96 = - 29

768223344=72344-62334+82234

= 7 (8 - 12) - 6(8 - 9) + 8 (8 - 6)

= 7 (- 4) - 6 (8 – 9) + 8 (8 - 6)

= - 28 + 6 + 16 = - 6

768222349=72249-62239+82234

= 7 (18 - 8) - 6 (18 - 6) + 8 (8 - 6)

= 7 (10) - 6 (12) + 8(2)

= 70 - 72 + 16 = 14

Substituting the above values in [P] we get,

ॐ Det (A) = 4(- 2) - 5(- 29) + 5(- 6) - 7(14) = 8 + 145 – 30 - 98 = 9

Evaluation of Determinant of Fourth order matrix by using vertically and crosswise Sūtra:

To find fourth order determinant by using vertically & crosswise Sūtra partitioning centrally horizontally, six second order determinants above the partition and the six second order determinants below the partition are obtained. The result of six second order determinants above the partition are placed above the determinant in a row, the result of six second order determinants below the partition are placed below the determinant in a row. The associated signs of both the new rows obtained from the six second order determinants are placed as below:

For the determinant of two adjacent columns '+' sign; for the determinant of the column with column one space apart '-' sign; for the determinant of the column with the column two space apart '+' sign.

The two outermost rows of figures are multiplied together according to the pattern:

-11 -3 -17 10 -2 -16

4 5 5 77 6 8 82 2 2 33 4 9 4

2 12 -1 10 -4 -19

By using Vertically & Crosswise Sutra,

+	-	+	+	-	+
-11	-3	-17	10	-2	-16
2	12	-1	10	-4	-19
+	-	+	+	-	+

Figure: 4.3

From the above figure with the help of V & C Sūtra,

∴det (A) = + (-11) (-19) - (-3) (-4) + (-17) (10) + (10) (-1) - (-2) (12) + (-6) (2)

= 209 – 12 – 170 – 10 + 24 – 32 = 9

Similarly, to evaluate the fourth order determinant by partitioning centrally vertically, six order determinants are obtained right side of the partitions and six order determinants are obtained left side of the partitions.

The above example can be solved and compared the result by partitioning centrally vertically with the result by partitioning centrally horizontally.

Evaluation of the determinant of 4×4 matrix by centrally vertically partitioning.

A=4557768822233494

A=det(A)= 4557768822233494 = -11 -2 +1 +2 +10 +2 4557768822233494-16 +1-

43 +8-40-19

By using Vertically & Crosswise Sūtra:

+ -11- -2+ +1+ +2- +10+ +2

-16 +1 -43 +8 -40 -19 + - + + - +

Figure: 4.4

From the above figure,

∴det(A)=(-11)(-19)-(-2)(-40)+(1)(8)+(2)(-43)-(10)(1)+2(-16)

=209-80+8-86-10-32=9

A pyramid layout of second–order sub-determinants:

(Centrally horizontal partition):

-17-3 -2-11 10 -164557768822233494 2 1 0 -

1912 -4-1

Figure: 4.5

Multiplication Pattern:

c = -17

b = -3 e = -2

a = -11…………d = 10………f = -16

p = 2…....s =10……u = -19

q =12 t= -4

r = -1

Figure: 4.6

∴ Det (A) = (-11) (-19) + (-16) (2) - (-3) (-4) - (-2) (12) + (-17) (10) + (10) (-1)

= 209 - 32-12 + 24 – 170 - 10 = 9

Also by partitioning vertically six second order determinants are obtained and find the determinant similarly.

4.3 Extraction of elements: (Expansion by individual elements):

The procedure using By Alternate Elimination and Retention Sūtra:

First eliminate the element, then everything else but that element and its expansion (the terms that go with it).

Eg: Extract the appropriate elements from:

[3]

X = 3425987214 (Solve it by extracting one element 98)

It can be written after extracting 98,

X = 342507214+983224 ……….. [I]

Where,

342507214=342507214

5 6 -7

For the first and last row by using Vertically & Crosswise Sūtra,

= 3(-7) – 4(6) + 2(5) = - 21 - 24 + 10 = -35

And

983224=9812-4=98(8)=784

By substituting both the values in [I]
The final answer = - 35 + 784 = 749

[4]

Y=58234216767 (Solve it by extracting two elements 58 and 67)

After extracting 58 & 67 given determinant can be written as,

Y=023421670+582170+670242+58×67×2 …… [II]

023421670=023421670

16 -6 -7

= 0 (- 7) - 2 (- 6) +3 (16) = 0 + 12 + 48 = 60

582170=58 (-7)=-406,

670242=67(-8)=-536 and 58×67×2=7772

Substituting the above values in [II],

Final answer = 60 + 58 (-7) + 67 (-8) + 58 × 67 × 2

= 60 - 406 - 536 + 7772 = 7832 – 942 = 6890

[5]

Z =48232699253211144331120 (Solve by extracting three elements 48, 99 and 111)

After extracting 48, 99 & 111

Z = 023260253204433120 + 48 02520433120 + 99 03230443120 + 111 02260543120

+ 48 × 99 04 3120 + 48 × 111 05 3120 + 99 × 111 024120

+ 48 × 99 × 111 × 120 …………………………………….. [III]

Where,

02520433120 = 02520433120 = 0 (-12) -2(228) + 5(6) = - 456 + 30 = - 426,

6   228   -12

03230443120 = 03230443120 = 0 (-12) – 3(344) + 2(9) = - 1032 +18 = -1014,

9   344   -12

02260543120 = 0 2260543120 = 0 (-15) – 2 (700) + 2 (18) = -1400 + 36 = - 1364,

18   700   -15

043120 = -12,   05 3120=-15,   024120 = - 8

The fourth order determinant can be obtained by using pyramid method by centrally horizontal partition.

-12
18   -10
-12   4   11

023260253204433120

1 6 -12
-9 -228
344
Figure: 4.7

From the above figure by using Vedic Vertically & Crosswise Sūtra,

∴ Det = (-12) (-12) + 11(1) - (-10) (-9) – (18) (-228) + (-12) (6) + 344 (4)

=144 + 11- 90 + 4104 – 72 + 1376
= 5635 – 162 = 5473

Final answer = 5473 + 48(-426) + 99(-1014) + 111(-1364) + 4752(-12)
+ 5328 (-15) + 10989(-8) + 63296640
= 5473 - 20448 - 100386 – 151404 – 57024 – 79920 – 87912 + 63296640
= 63302113 – 497094
= 62805019

A special case:

A special case comes under Sunyam Samyasmuccaye Sūtra
[6]

Solve: D = 491162 16 9213

D = 4911 62 16 9213 = 494+9+11626+2+1692 9+2+13

= 492462 24 9224 = 24 49162 1 921 = 24(21) = 504,

∵ 491621921 = 491621921=40-9-3+1-6=27-6=21

-6 -3 0

4.4 Applications of Determinants:

4.4.1 Use of Determinant in solving Simultaneous Linear Equations:

[7]

Solve: 3x + 2y = 5 ……. (1)
4x + y = 3....……. (2)

In “Vedic Mathematics” the following method is given for solving two equations in two unknowns.

x=2.3-5.14.2-3.1=15 & y=5.4-3.34.2-3.1=115

By using Cramer’s Rule,

x=-25133241=-1-5=15 & y=35433241=-11-5=115

Another Method:

By using equations (1) and (2), we get two rows of a third order determinant

325413

From this three second order determinant give, on sorting out the signs, x .D, y .D and D, where

D=3241

x. D is given by the determinant arising on missing out the first column, -y .D from missing out the second column and –D from missing out the third. This leads us to consider the third order determinant:

det A=x.D-y.D-D325413

On evaluating the determinant the coefficient of D, x .D and y. D give the respective value of D, x. D and y. D Thus, Det A = x. D + 11y. D + 5D

Hence, D equals five and the product of x & y with D has values of unity & eleven respectively. Whence x = 1/5 and y = 11/5.

These procedure we can also apply for solving ‘n’ equations in ‘n’ unknowns.

[8]

Solve the following simultaneous linear equations:

$$3x + 2y + z = 10$$
$$5x + 3y + 2z = 17$$
$$7x + 8y + z = 26$$

Consider the fourth order determinant

x.D	y. D	z. D	-D		
-1	3	2	1	10	-3
-10	5	3	2	17	-16
19	7	8	1	26	35

For finding D: Covering the fourth column, we evaluate the 3 by 3 determinant of the remaining three columns using the cross-product procedure. We have

-1 1

-10 2

19 1

Figure: 4.8

i.e. $-1 \times 1 + -10 \times 2 + 19 \times 1 = -1 - 20 + 19 = -2$

To evaluate x. D cover the first column and evaluate the 3 by 3 determinant of the remaining three columns using the cross-product procedure.

2 -3

3 -16

8 35

Figure: 4.9

i.e. 2 × 35 + 3 × (-16) + 8 × (-3) = 70 – 48 - 24 = -2

To evaluate -y. D cover the second column and evaluate the 3 by 3 determinant of the remaining three columns using the cross-product procedure.

3 -3

5 -16

7 35

Figure: 4.10

i.e. 3 × 35 + 5 × (-16) + 7 × (-3) = 105 - 80 – 21 = 4

To evaluate z. D cover the third column and evaluate the 3 by 3 determinant of the remaining three columns using the cross-product procedure.

-1 10

-10 17

19 26

Figure: 4.11

i.e. (-1) × 26 + (-10) × 17 + 19 × 10 = - 26 - 170 + 190 = - 6

Thus D = -2, x. D = - 2, -y. D = 4 and z. D = - 6.

∴ The final solution is x = 1, y = 2 & z = 3

Procedure for eliminating one variable or two variables at a time:

Determinants arise on eliminating unknowns from simultaneous equations: second order determinant on eliminating one unknown, and third order on eliminating two unknowns, etc.

[9]

Solve the following:

$4x + 8y = 20$

$2x + 5y = 12$

Eliminating variable x: Eliminating variable y:

4825y=420212 -4825x=820512∴(20-16)y=(48-40) ∴-(20-

16)x=(96-100)∴4y=8∴y=2 ∴-4x=-4∴x=1

Ans. [$x = 1$, $y = 2$]

[10]

Solve the following:

$4x + 2y + 2z = 22$…….. (1)

$2x + 4y + 7z = 38$…….. (2)

$8x + 4y + 2z = 36$…… (3)

Eliminating x from (1) and (2)

4224y+4227z=422238

∴(16-4)y+(28-4)z=(152-44)

∴12y+24z=108…………(4)

Eliminating y from (1) and (2)

- 4224x+2247z=222438

∴-(16-4)x+(14-8)z=(76-88)

∴-12x+6z=-12…….(5)

Eliminating x from (2) and (3)

2   48   4y+2   78   2z=238836

$$\therefore (8-32)y+(4-56)z=(72-304)$$

$$\therefore -24y-52z=-232\ldots\ldots(6)$$

Solving equations (4), (5) and (6)
We get x = 3, y = 1, z = 4.

4.4.2 Non zero solution of Linear Homogeneous Equation:

The homogeneous equation in x, y, z are
$a_1 x + a_2 y + a_3 z = 0$
$b_1 x + b_2 y + b_3 z = 0$
$c_1 x + c_2 y + c_3 z = 0$
A system of simultaneous equations have non-zero value or trivial solutions if all the unknowns have zero values, and is said to have non-zero solution if at least one of the unknown has the non-zero value.

The necessary and sufficient condition that the equations have non-zero solution is

$$a1a2a3b1b2b3c1c2c3=0$$

[11]

Test whether the following equations have non-zero solution. If they have such solution, obtain the solutions.

$x + 2y - 3z = 0$
$3x - 2y - z = 0$
$2x + 2y - 4z = 0$

The necessary and sufficient condition that the equations have non-zero solution is

1 2-33-2-12 2-4=0

Here,1 2-33-2-12 2-4 =1 2-33-2-12 2-4

10 -10 10

By using Vertically & Crosswise Sūtra,

∴ Det = 1(10) – 2(-10) – 3 (10) = 0 and hence the equations have non-zero solution.

Solving first two equations,

$x + 2y - z = 0$

$3x - 2y - z = 0$

$\therefore x-8=-y-8=z-8$

So that $x=-8\lambda, y=-8\lambda, z=-8\lambda$; where λ is a non-zero solutions.

These value satisfy the third equation,

$2x + 2y - 4z = 2(-8\lambda) + 2(-8\lambda) - 4(-8\lambda)$

$= -16\lambda - 16\lambda + 32\lambda = 0.$

Hence, they are non-zero solution.

[12]

Solve:

2u + 3v + 4w = 0

u – 2v – 3w = 0

3u + v – 8w = 0

The necessary and sufficient condition that the equations have non-zero solution is 23 41-2-

331-8=0

23 41-2-331-8 = 2 3 41-2-33 1-8…………………………… 7 1 19

By using Vertically & Crosswise Sūtra,

Det =2(19) - 3(1) + 4(7) = 38 – 3 + 28 = 63 ≠ 0 and

Hence, the equations have no non-zero solution.

u = v = w = 0

4.5 Adjoint of a Matrix:

Minors:

Consider the determinant $A=a11a12a13a21a22a23a31a32a33$

If the element a_{ij} leave the row and column passing through the element a_{ij}.
Minor of is a_{ij} is written as M_{ij}.

The minor of the element $a12=a21a23a31a33=M12$

The minor of the element $a23=a11a12a31a32=M23$

The minor of the element $a33=a11a12a21a22=M33$

Co-factors:

The minor multiply by $(-1)^{i+j}$ is called the cofactor of the element a_{ij}.

The Co-factor of the element a12 = A12

A12=-11+2M12

=(-1)1+2a21a23a31a33=(-1)3a21a23a31a33=-a21a23a31a33

The Co-factor of the element a23=A23

A23=-12+3M23

=(-1)2+3a11a12a31a32=(-1)5a11a12a31a32=-a11a12a31a32

The Co-factor of the element a33= A33

A33=-13+3M33

=(-1)6a11a12a21a22=a11a12a21a22

Adjoint of a Matrix:

Adjoint of a Matrix is obtained by first finding a matrix formed by the co-factors of the elements of the given matrix A and then by taking transpose of it.

If A=a11a12a13a21a22a23a31a32a33 then

Adj (A) = transpose of the matrix formed by co-factor

Adj(A)=A11A21A31A12A22A32A13A23A33 Where,Aij is the Co-factor of the

element aij.

[13]

If A=133143134is given matrix, then find its Adjoint.

A11 = the co-factor of a11in

A = (-1)1+14334=(-1)24334=4334=16-9=7

A12= the co-factor of a12in

A = (-1)1+21-323=(-1)31-323=-1-323=-(3+6)=-9

Similarly,

A13= -5, A21= - 4, A22= 1, A23= 3,A31= -5, A32= 4,A33= 1

Adj (A) = transpose of the matrix formed by co-factor

Adj(A)=A11A21A31A12A22A32A13A23A33=7-4-5-914-531

Where, Aijis the Co-factor of the element aij.

∴Adj(A)=7-4-5-914-531

4.6 Square inverse matrices:

It is mandatory for a matrix to have any determinant value other than zero so as to have its inverse in mathematical existence a possibility. Therefore, all matrices must fulfill this constraint of non singularity.

∴A-1=adjAA;A≠0

[14]
Find Inverse of the matrix of order 3x3 by finding its adjoint

B = 491621921

B= 491621921

-6 -3 0

By V & C Sūtra from the figure above,

∴ Det B = 4 (0) – 9 (- 3) +1(- 6) = 27 - 6 = 21 which is non-zero.

Since B≠0, B is non-singular.Therefore, Inverse of B exits.

For finding co-factor of B, find all the minors of matrix B,

2121=10=0

Minor of 4 = $(-1)^{1+1}$

6191=-16-9=-1-3=3

Minor of 9 = $(-1)^{1+2}$

6292=112-18=-6

Minor of 1 = $(-1)^{1+3}$

9121=-19-2=-17=-7

Minor of 6 = $(-1)^{2+1}$

4191=14-9=1-5=-5

Minor of 2 = $(-1)^{2+2}$

4992=-18-81=-1-73=73

Minor of 1 = $(-1)^{2+3}$

9121=19-2=17=7

Minor of 9 = $(-1)^{3+1}$

4161=-14-6=-1-2=2

Minor of 2 = $(-1)^{3+2}$

4962=18-54=-46

Minor of 1 = $(-1)^{3+3}$

Co-factor of B = 03-6-7-57372-46

Transforming rows into columns of above matrix Adjoint matrix can be obtained as follows:

Adjoint of B = 0-773-52-673-46

∴B-1=adjBB;B≠0

∴B-1=1210-773-52-673-46

[15]

Find Inverse of the matrix of order 4x4 by finding its adjoint

A=a11a12a13a14a21a22a23a24a31a32a33a34a41a42a43a44=1001021221012014

A=det(A)=1001021221012014

By using Pyramid method by centrally horizontal partition very easily.

2
1 -2
2 0 -1

1001021221012014

-2 1 -1
2 4
6

Figure: 4.12

From the figure with the help of Vertically & Crosswise Sūtra,

∴det (A) = 2 (-1) + (-1) (-2) + 1(4) + (-2) (2) + 2 (1) + 0 (6)

= -2 + 2 + 4 – 4 + 2 = 2

To find co-factor of A, taking minors of all the elements of matrix A

212101014

Minor of a_{11}= 1 = $(-1)^{1+1}$ = $(-1)^2$ [2 (-1) -1(4) + 2(1)] = -2 – 4 + 2 = - 4

1 4 -1

012201214

Minor of a_{12} = 0 = $(-1)^{1+2}$ = $(-1)^3$ [0(-1)-1(6) + 2(2)] = - [0 - 6 + 4] = 2

2 6 -1

022211214

Minor of a_{13} = 0 = $(-1)^{1+3}$ = $(-1)^4$ [0 - 2(6) + 0] = -12

0 6 3

021210201

Minor of a_{14} = 1= $(-1)^{1+4}$ = $(-1)^5[0 - 2(2) + 1(-2)] = - [- 4 - 2] = 6$

-2 2 2

001101014

Minor of a_{21} = 0 = $(-1)^{2+1}$ = $(-1)^3[0 + 0 + 1(1)] = -1$

1 4 -1

101201214

Minor of a_{22} = 2 = $(-1)^{2+2}$ = $(-1)^4[1(-1) - 0(6) + 1(2) = -1- 0+2 = 1$

2 6 -1

101211204

Minor of a_{23} = 1= $(-1)^{2+3}$ = $(-1)^5[1(4) – 0 + 1(-2)] = - [4 – 0 - 2] = -2$

-2 6 4

100210201

Minor of a_{24} = 2 = $(-1)^{2+4}$ = $(-1)^6 [1(1) - 0 + 0] = 1$

-2 2 1

001212014

Minor of $a_{31} = 2 = (-1)^{3+1}$ $= (-1)^4 [0 – 0 + 1(2)] = 2$

2 8 2

101012214

Minor of $a_{32} =1 = (-1)^{3+2}$ $= (-1)^5 [1(2) – 0 + 1(-2)] = - [2 - 2] = 0$

-2 - 4 2

101022204

Minor of $a_{33} = 0 = (-1)^{3+3}$ $= (-1)^6 [1(8) – 0 + 1(-4)] = 8 - 4 = 4$

-4 -4 8

100021201

Minor of $a_{34} = 1 = (-1)^{3+4}$ $= (-1)^7 [1(2) - 0 + 0] = -2$

-4 -2 2

001212101

Minor of a_{41} = 2 = $(-1)^{4+1}$ = $(-1)^{5}$ [0 – 0 + 1(-1)] = - (-1) =1

-1 0 1

101012201

Minor of a_{42} = 0 = $(-1)^{4+2}$ = $(-1)^{6}$ [1(-2) – 0 + 1(-2)] = -2 - 2 = - 4

-2 - 4 1

101022211

Minor of a_{43} = 1= $(-1)^{4+3}$ = $(-1)^{7}$ [1(0) - 0 + 1(-4)] = - (-4) = 4

-4 - 4 0

100021210

Minor of a_{44} = 4 = $(-1)^{4+4}$ = $(-1)^{8}$ [1(-1) – 0 + 0] = -1

-4 -2 -1

Co-factors of A= -42-126-11-21204-21-44-1

Adjoint of A = -4-121210-4-12-24461-2-1

A-1=adj AA

Also determinant of A is 2.

∴A-1=12-4-12   1 2   10-4-12-24 4   6 1-2-1

4.7 Rank of Algebraic Equation:

Sub Matrix:

Sub matrix of any matrix is obtained by omitting specific vertical & horizontal number arrangements from original matrix of order m x n.

E.g. 6 04 4-2 148 18 14-140-10

Contains four 3rd order sub matrices,

604-2 148 14-140, 04 4 148 18-140-10, 64 4-28 18 140-10, 604-21418

14-14-10

Contains fifteen 2rd order sub matrices,

60-214,6014-14,-21414-14,04148,04-140,

148-140,44818, 440-10,8180-10,64-28,

64140,-28140,6414-10,64-218,-21814-10

Contains three row matrix 6044,-214818,14-140-10

Contains column matrix 6-214,014-14,480,418-10 and so on.

Rank of a Matrix:

We can say that the rank of a matrix is r if,
When starting from smallest sub matrices, all the determinants result in trivial solution up until the sub matrix of order r, beyond which determinant of higher sub matrices are of non trivial significance.
[16]
Evaluate rank:

604   4-214 8   1814-14 0-10

The matrix is of order 3 x 4.

Hence, rank of the given matrix i.e. ρ(A) ≤ 3.

The matrix contains four 3x3 matrices

6 04-2 148 14-140; 04 4 148 18-140-10;

644-28 18 140-10; 60 4-2 14 18 14-14-10

The determinant of all these are zero.

Hence, rank of the given matrix is less than or 2.i.e.ρ(A)≤2

60-214=84≠0 ∴ρ(A)=2

∴Rank of the given matrix is 2.

4.8 Consistency of Linear Equations:

Consider a set of equations

$a_1x + a_2y + a_3z = d_1$

$b_1x + b_2y + b_3z = d_2$

$c_1x + c_2y + c_3z = d_3$.

The matrix arrangement of these three relationships are displayed:

a1a2a3b1b2b3c1c2c3xyz=d1d2d3

If we join the matrices i.e. [A:D]=a1a2a3b1b2b3c1c2c3: :: d1d2d3.

It is called as Augmented Matrix. Reduce [A: D] to Echelon form, and find rank of A and rank of [A: D].

If $\rho(A)\neq\rho(AD)$,then result cannot be obtained.

If $\rho(A)=\rho(AD)$, then the system is consistent and

If $\rho A=\rho AD$ = No. of unknowns we can say that all equations are consistent & its final result must be distinct.

If $\rho A=\rho AD<$ Number of unknowns then the system is consistent and has infinitely many solution.

[17]
Discuss the consistency of

$x + y + z = 1,$

$2x + 4y - 3z = 9,$

$3x + 5y - 2z = 11$

In the matrix form

A=11 124-335-2 and [A:D]=11124-335-2 ::: 1 911

First we will find Rank of A,

Since11 124-335-2 = 1 1 124-335-2 = 1(7) - 1(5) + 1(-2) = 7 – 5 - 2 = 0

-2 5 7

i.e. 11 124-335-2 = 0.

Therefore, Rank (A) must be less than or equal to two.

1124=4-2=2≠0∴ρ(A)=2

Now find rank of augmented Matrix [A: D]

[A:D]=11 1 24-3 35-2 :::19 11

Since 1114-395-211 =1 114-395-211 = 1(-15) - 1(-1) + 1 (7) = 7 ≠ 0

$-15 \quad -1 \quad 7$

$\therefore \rho A:D=3$

but $\rho(A)=2$

$\therefore \rho(A) \neq \rho[A:D]$

Thus, the system is inconsistent .i.e. it has no solution.

[18]

Discuss the consistency of

$5u + 3v + 7w = 4$

$3u + 26v + 2w = 9$

$7u + 2v + 10w = 5$

The above equations can be written in the matrix form

A=53732627210 & [A:D]=53732627210:::495

First we will find Rank of matrix A =53732627210

Since53732627210 = 53732627210 = 5(256) - 3(16) + 7(-176) = 0

-176 16 256

i.e. 53732627210 = 0.

Therefore, Rank (A) must be less than or equal to two.

262210=260-4=256≠0∴ρ(A)=2

Now find rank of augmented Matrix [A: D]

$A:D=53732627210::: \ 4 \ 9 \ 5$

Contains four 3rd order sub - matrices,

$53732627210 \ , 5743297105,5343269725,37426292105$

All 4 sub - matrices have determinant zero. $\therefore \rho[A:D] \leq 2$

But, $262210=260-4=256 \neq 0 \therefore \rho[A:D]=2$

$\therefore \rho[A:D]=\rho(A)=2$

$\therefore \rho A=\rho A:D=2<3=$ Number of unknown

Thus the system is consistent.

I.e. it has infinitely many solutions.

[19]

Discuss the consistency of

$3u + v + 2w = 3$

$2u – 3v – w = -3$

$u + 2v + w = 5$

In the matrix form

A=3   1 22-3-11   2 1 And [A:D]=31 22-3-112 1:::   3-3   5

First for finding Rank of A.

Since 3   122-3-11   21= 3   122-3-11   2   1

7   3   -1

By using V & C Sutra for 1st & last row,

det (A) = 3 (-1) - 1 (3) + 2 (7) = 8 $\neq 0$

Therefore, Rank (A) must be 3.

Now find rank of augmented Matrix [A: D]

[A:D]=3   1 22-3-11   2 1::: 3-3   4

Since the above augmented matrix contain four 3rd order Sub-matrices in order to find out rank, the determinants of all sub-matrices are obtained as follows:

3   1   22-3-11 2   1=3   1   22-3-11 2   1=3-1-13+27=-3-3+14=8≠0

7   3   -1

3   1   32-3-31 2   4=3   1   32-3-31 2   4=3-6-111+37=-8≠0

7   11   -6

3   2   32-1-31 1   4= 3   2   32-1-31 1   4=3-1-211+33=-16≠0

3   11   -1

1   2   3-3-1-32 1   4=1   2   3-3-1-32 1   4 1-1-2-6+3-1=8 ≠0

-1   -6   -1

∴ρA:D=3∴ρA=ρA:D=3= Number of unknowns

Thus, the system is consistent .i.e. it has unique solution.

Chapter: 5

Vedic mathematics in Derivatives and Integration, Differential Equations and Partial Differential Equations

5.1 Introduction to Differential Calculus:

The most important achievement in the mathematics is the development of both the Differential Calculus and the Integral Calculus. There are lots of real life applications which require us to find the rate of change of one parameter with respect to another for which Differential Calculus plays an important role.

In order to find the derivative of any function at a given point geometrically, draw a tangent at that point and evaluate the value of the slope. If y = f(x) given then

$$\frac{dy}{dx} = f'(x).$$

By using Dhvaja Ghata (power):

The method to find first differential of the each term of the quadratic expression $ax^2 + bx + c$, is by multiplying its Dhvaja Ghata (power) by the Anka (its coefficient) and reducing by one.

Example: 1

Find derivative of quadratic expression $x^2 - 9x + 14$.

Let $E = x^2 - 9x + 14$.

As per **current method**, taking derivative of y w.r.t. x

$$\therefore \frac{dy}{dx} = f'(x) = \frac{d}{dx}(x^2) - 9\frac{d}{dx}(x) + \frac{d}{dx}(14)$$

$$= 2x - 9(1) + 0$$

$$\therefore \frac{dy}{dx} \quad = 2x - 9$$

By using **Dhvaja Ghata,**

Finding first differential of each term of quadratic expression $x^2 - 9x + 14$,

x^2 gives 2x; - 9x gives - 9 and 14 gives zero.

Therefore, $D_1 = f'(x) = \frac{d}{dx}(x^2 - 9x + 14) = 2x - 9$.

Calana-Kalanābhyām Sūtra:

Meaning: Differential Calculus

This Sūtra shows the relationship between first differential and square root of Discriminant of the quadratic. ŚRĪ BHĀRATĪ KṚṢṆA called this Sūtra as a **Calculus formula** for finding two roots of given quadratic equation. According to him the square root of the discriminant of original quadratic equation is equal to the first differential.

i.e. $D_1 = \pm\sqrt{\text{Discriminant}} = \pm\sqrt{b^2 - 4ac}$

Thus, roots of given quadratic equations are obtained by solving two simple equations.

Example: 2

Solve the quadratic equation $x^2 - x - 12 = 0$

The first differential $D_1 = 2x - 1$ and

The square root of the discriminant is $\pm\sqrt{1 + 48} = \pm\sqrt{49} = \pm 7$

As per above rule, $D_1 = \pm\sqrt{\text{Discriminant}} \therefore 2x - 1 = \pm 7$

Thus the given equation is broken down into two simple equations

$\therefore 2x - 1 = \pm 7$

$\therefore 2x = \pm 7 + 1$

$\therefore\ x = 4 \text{ OR } x = -3$

5.2 Factors of Polynomials & Differential Calculus by Guṇakasamuccayaḥ :

In polynomials, there is a relationship between factors & its successive differentials. If one knows factors of given polynomial one can derive its differentials.

Procedure for finding Differentials:

Write the given polynomial in its standard form with 1 being the constant term preceding x^n.

Then break down the polynomial into linear factors.

According to Guṇakasamuccayaḥ Sūtra, if a polynomial is the product of factors, then by adding that factors 1st differential D_1 of given polynomial is obtained. The procedure of finding Differentials of quadratic Expression, Cubic Expression and Quartic Equation and also for the polynomial with 5th Power with examples are explained below.

D_5 = 5! [Sum of factor-products taken (n-5) at a time]

5.2.1 Differentials of quadratic expression:

According to Guṇakasamuccayaḥ Sūtra, if we can factorize the quadratic equation in two linear factors $a_1 = x + m$and $a_2 = x + n$ then then by adding that two linear factors we get 1st differential D_1.

First Differential D_1 = 1! [Sum of factor product taken (2 - 1) = 1 at a time]

i.e. $D_1 = \sum_{i=1}^{2} a_i = a_1 + a_2 = (x + m) + (x + n)$

Second differential D_2 = 2!

Example: 3

Evaluate possible differentials for given expression $x^2 - 9x + 14$

As per current method,

$D_1 = \frac{d}{dx}(x^2 - 9x + 14) = 2x - 9$

$D_2 = \frac{d}{dx}(2x - 9) = 2 = 2!$

Let E = $x^2 - 9x + 14$, then factors of quadratic expression E = $(x - 2)(x - 7)$

According to Guṇakasamuccayaḥ Sūtra,

First differential $D_1 = \sum_{i=1}^{2} a_i = (x - 2) + (x - 7) = 2x - 9$

Second Differential D_2 = 2!

5.2.2 Differentials of Cubic Expression:

For cubic expression if we can factorize into three linear factors

$$a_1 = (x + r), \qquad a_2 = (x + s) \text{ and } a_3 = (x + t);$$

Now, differentials are as follows.

First Differential D_1 = 1! [Sum of factor product taken (3-1) = 2 at a time]

i.e. $D_1 = \sum_{i,j=1}^{3} a_i\, a_j = [\, a_1a_2 + a_2a_3 + a_1a_3\,]; i < j,\ i \neq j$

$$= [(x + r)(x + s) + (x + s)(x + t) + (x + r)(x + t)]$$

Second Differential D_2 = 2! [Sum of factor product taken (3 - 2) = 1 at a time]

i.e. $D_2 = 2! \sum_{i=1}^{3} a_i = 2!\,[\, a_1 + a_2 + a_3\,] = 2!\,[(x + r) + (x + s) + (x + t)]$

Third differential D_3 = 3!

Example: 4

Find possible number of differentials of cubic expression $x^3 + 7x^2 + 14x + 8$.

First find factors of $E = x^3 + 7x^2 + 14x + 8$

Sum of all coefficient is not zero.i.e.$1 + 7 + 14 + 8 \neq 0$

$\therefore$ $(x - 1)$ is not a factor of given expression.

Sum of co-efficient of even power of $x = 7 + 8$

Sum of co-efficient of odd power of $x = 1 + 14$

I.e. $7 + 8 = 1 + 14 = 15$

$\therefore$ One factor of given expression is $(x + 1)$.

By Ādyamadyenāntyamantyena, divide the 1st and the last term of given expression E by the first and last term of already found factor i.e. $(x + 1)$ respectively.

Since given expression is in cubic form, we get the first and last co-efficient of the quadratic expression.

$$\text{First co-efficient} = \frac{x^3}{x} = x^2 \quad \& \ \text{last co-efficient} = \frac{8}{1} = 8$$

For finding middle coefficient, subtract the sum of the above obtained co-efficient of quadratic expression

I.e. $1 + 8 = 9$ from the sum of co-efficient of odd (even) power $= 15$

i.e. $15 - (1 + 8) = 15 - 9 = 6$ is the co-efficient of the middle term of quadratic expression.

$\therefore Q = x^2 + 6x + 8$

By using Ānurūpyeṇa and Ādyamadyenāntyamantyena Sūtras:

$Q = (x + 2)(x + 4)$

$\therefore E = x^3 + 7x^2 + 14x + 8 = (x + 1)(x + 2)(x + 4)$

Another way for finding Factors:

Let $E = x^3 + 7x^2 + 14x + 8$

Here,

Sum of co-efficient of even power of x = Sum of co-efficient of odd power of $x = 15$

& the last term is 8.

Factors of 8: 1, 2, 4 OR 2, 2, 2.

Sum of the three factors = 7 (∵ co-efficient of $x^2 = a + b + c = 7$)

∴ We select the group 1, 2, 4. [∵ $1 + 2 + 4 = 7$]

Also testing for the co-efficient of x.

Here $(a \times b + b \times c + c \times a) = 14$ = co-efficient of x.

∴ $E = x^3 + 7x^2 + 14x + 8 = (x + 4)(x + 2)(x + 1)$

First differential $D_1 = \sum_{i,j=1}^{3} a_i a_j$; $i < j,\ i \neq j$

$= (x+4)(x+2) + (x+2)(x+1) + (x+1)(x+4)$

$= x^2 + 6x + 8 + x^2 + 3x + 2 + x^2 + 5x + 4$

$= 3x^2 + 14x + 14$

Second differential $D_2 = 2! \sum_{i=1}^{3} a_1$

$= 2\,[(x+4) + (x+2) + (x+1)]$

$= 2\,(3x + 7) = 6x + 14$

Third differential $D_3 = 3! = 6$

Example: 5

Find the possible differentials of cubic expression $x^3 - 6x^2 + 5x + 12$

First find the factors of the given cubic expression.

Let the factors of $E = x^3 - 6x^2 + 5x + 12$ be $(x + 1)$, $(x - 3)$ & $(x - 4)$

First differential $D_1 = \sum_{i,j=1}^{3} a_i a_j$; $i < j,\ i \neq j$

$= [(x + 1)(x - 3) + (x - 3)(x - 4) + (x - 4)(x + 1)]$

$= x^2 - 2x - 3 + x^2 - 7x + 12 + x^2 - 3x - 4$

$= 3x^2 - 12x + 5$

Second differential $D_2 = 2! \sum_{i=1}^{3} a_i$

$= 2\,[(x + 1) + (x - 3) + (x - 4)]$

$= 6x - 12$

Third differential $D_3 = 3! = 6$

5.2.3 Differentials of bi-quadratic expression:

For bi-quadratic expression we can factorize them into four linear factors

$$a_1 = (x + m), \qquad a_2 = (x + n), \qquad a_3 = (x + p) \text{ and } a_4 = (x + q)$$

Thus, First Differential $D_1 = 1!$ [Sum of factor product taken (4-1) = 3 at a time]

$$\text{i.e. } D_1 = \sum_{i,j=1}^{4} a_i a_j a_k; i < j < k, \ i \neq j \neq k$$

$$= a_1a_2a_3 + a_1a_2a_4 + a_2a_3a_4 + a_1a_3a_4$$

$$= (x + m)(x + n)(x + p) + (x + m)(x + n)(x + q)$$

$$+ (x + n)(x + p)(x + q) + (x + m)(x + p)(x + q)$$

Second Differential $D_2 = 2!$ [Sum of factor product taken (4-2) = 2 at a time]

$$\text{i.e. } D_2 = 2! \sum_{i,j=1}^{4} a_i a_j; i < j, \ i \neq j$$

$$= 2! [\, a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4 \,]$$

$$= 2! [(x + m)(x + n) + (x + m)(x + p) + (x + m)(x + q)$$

$$+ (x + n)(x + p) + (x + n)(x + q) + (x + p)(x + q)$$

Third Differential $D_3 = 3!$ [Sum of factor product taken (4-3) = 1 at a time]

$$\text{i.e. } D_3 = 3! \sum_{i=1}^{4} a_i$$

$$= 3! [\, a_1 + a_2 + a_3 + a_4 \,]$$

$$= 3! [(x + m) + (x + n) + (x + p) + (x + q)]$$

Fourth Differential $D_4 = 4!$

Example: 6

Find the possible differentials of bi-quadratic expression $x^4 - 8x^3 + 8x^2 + 32x - 48$

First find factors of given bi-quadratic expression.

Let the factors of $E = x^4 - 8x^3 + 8x^2 + 32x - 48$ be (x - 2), (x - 2), (x + 2) & (x - 6)

First differential $D_1 = \sum_{i,j,k=1}^{4} a_i a_j\, a_k;\, i < j < k,\ i \neq j \neq k$

$$D_1 = [(\ x + 2\ )(\ x - 2\ )(\ x - 2\ ) + (\ x - 2\ )(\ x - 2\ )(\ x - 6\ )$$
$$+ (x + 2)(x - 2)(x - 6) + (x + 2)(x - 2)(x - 6)]$$

$$= [\ x^3 - 2x^2 - 4x + 8 + x^3 - 10x^2 + 28x - 24 + x^3 - 6x^2 - 4x + 24 + x^3 - 6x^2 - 4x + 24\ ]$$

$\therefore D_1 = 4x^3 - 24x^2 + 16x + 32$

Second differential $D_2 = 2! \sum_{i,j=1}^{4} a_i\, a_j;\, i < j,\ i \neq j$

$$\therefore D_2 = 2!\ [(x + 2)(x - 2) + (x + 2)\ (x - 2) + (x + 2)\ (x - 6)$$
$$+ (x - 2)(x - 2) + (x - 2)(x - 6) + (x - 2)(x - 6)]$$

$\therefore D_2 = 2!\ [x^2 - 4 + x^2 - 4 + x^2 - 4x - 12 + x^2 - 4x + 4 + x^2 - 8x + 12 + x^2 - 8x + 12]$

$= 2\ [6x^2 - 24x + 8]$

$\therefore D_2 = 12x^2 - 48x + 16$

Third differential $D_3 = 3!\ \sum_{i=1}^{4} a_i$

$$= 3!\ [(x + 2) + (x - 2) + (x - 2) + (x - 6)]$$

$\therefore D_3 = 6(4x - 8) = 24x - 48$

Fourth differential $D_4 = 4! = 24$

5.2.4 Differentials of fifth power of x:

Guṇakasamuccayaḥ Sūtra can be applied to solve the given equation with the greatest power of variable being of the fifth order, thereby factoring the expression into five sub parts as follows:

First Differential $D_1 = 1!$ [Sum of factor product taken (5 - 1) = 4 at a time]

$$D_1 = \sum_{i,j,k=1}^{5} a_i a_j\, a_k a_l;\, i < j < k < l,\ i \neq j \neq k \neq l$$
$$= [a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5]$$

$$\therefore D_1 = [(x + m)(x + n)(x + p)(x + q) + (x + m)(x + n)(x + p)(x + r)$$
$$+ (x + m)(x + n)\ (x + q)(x + r) + (x + m)(x + p)(x + q)(x + r)$$
$$+ (x + n)\ (x + p)\ (x + q)\ (x + r)]$$

Second Differential D_2 = 2! [Sum of factor product taken (5 - 2) = 3 at a time]

$$D_2 = 2!\sum_{i,j=1}^{5} a_i\, a_j a_k; i < j < k,\ \ i \neq j \neq k$$

$\therefore D_2 = 2[(x + m)(x + n)(x + p) + (x + m)(x + n)\ (x + q) + (x + m)\ (x + n)$

$(x + r) + (x + n)\ (x + p)(x + q) + (x + n)\ (x + p)\ (x + r) + \ldots\ldots\ldots\ldots]$

Third Differential D_3 = 3! [Sum of factor product taken (5 - 3) = 2 at a time]

$$D_3 = 3!\sum_{i=1}^{5} a_i a_j\, ; i < j,\ \ i \neq j$$

$\therefore D_3 = 6[(x + m)\ (x + n) + (x + m)\ (x + p) + (x + m)\ (x + q)$

$+ (x + m)(x + r) + (x + n)\ (x + p) + (x + n)\ (x + q) + (x + n)\ (x + r) + \ldots\ldots]$

Fourth Differential D_4 = 4! [Sum of factor product taken (5 - 4) = 1 at a time]

$$D_4 = 4!\sum_{i=1}^{5} a_i$$

$\therefore D_4 = 24\ [(x + m) + (x + n) + (x + p) + (x + q) + (x + r)];$

Fifth Differential $D_5 = 5!$

Example: 7

Find the possible differentials of the given expression

$E = x^5 + 20x^4 + 155x^3 + 580\ x^2 + 1044x + 720$

First find factors of given expression.

Let $E = x^5 + 20x^4 + 155x^3 + 580\ x^2 + 1044x + 720$

& factors of E are (x + 2), (x + 3), (x + 4), (x + 5) and (x + 6)

First differential $D_1 = \sum_{i,j,k=1}^{5} a_i a_j\, a_k a_l; i < j < k < l,\ \ i \neq j \neq k \neq l$

$\therefore D_1 = [(x + 2)(x + 3)(x + 4)(x + 5) + (x + 2)(x + 3)(x + 4)(x + 6)$

$$+(x+2)(x+3)(x+5)(x+6)+(x+2)(x+4)(x+5)(x+6)$$

$$+(x+3)(x+4)(x+5)(x+6)]$$

$\therefore D_1 = 5x^4 + 80x^3 + 465\,x^2 + 1160x + 1044$

Second differential $D_2 = 2! \sum_{i,j=1}^{5} a_i\, a_j a_k ;\ i < j < k,\ i \neq j \neq k$

$$\therefore D_2 = 2\,[(x+2)(x+3)(x+4)+(x+2)(x+3)(x+5)+(x+2)(x+3)(x+6)$$

$$+(x+3)(x+4)(x+5)+(x+3)(x+4)(x+6)+\cdots]$$

$\therefore D_2 = [20x^3 + 240\,x^2 + 930x + 1160]$

Third differential $D_3 = 3! \sum_{i=1}^{5} a_i a_j\,; i < j,\ i \neq j$

$$= 6[(x+2)(x+3)+(x+2)(x+4)+(x+2)(x+5)+(x+2)(x+6)$$

$$+(x+3)(x+4)+(x+3)(x+5)+(x+3)(x+6)+\ldots]$$

$\therefore D_3 = [60\,x^2 + 480x + 930]$

Fourth differential $D_4 = 4!\ \sum_{i=1}^{5} a_i$

$$= 24\,[(x+2)+(x+3)+(x+4)+(x+5)+(x+6)]$$

$\therefore D_4 = 24[5x + 20] = [120x + 480]$

Fifth differential $D_5 = 5! = 120$

Successive differentials and Factorization:

There is a close relationship between Factorization and Differentiation when considering the application of continuous derivative in order to identify the factors which are trivial.

Example: 8

Factorize: $x^4 - 8x^3 + 8x^2 + 32x - 48$

$\therefore D_1 = 4x^3 - 24x^2 + 16x + 32$

$= 4(x^3 - 6x^2 + 4x + 8)$

$= 4\,(x - 2)\,(x^2 - 4x - 4)$

∴ (x-2) is one factor.

∴$D_2 = 12x^2 - 48x + 16 = 4(3x^2 - 12x + 4)$ which cannot be factorize.

∴ $D_3 = 24x - 48 = 24(x - 2)$

∴We have $(x - 2)^2 = x^2 - 4x + 4$

To find another quadratic, $x^4 - 8x^3 + 8x^2 + 32x - 48 = (x^2 - 4x + 4)$ (…………..)

By using Vertically & Crosswise Sūtra for division,

$$\begin{array}{ccc} x^2 & -4x & +4 \\ x^2 & mx & -12 \\ \hline x^4 \quad - 8x^3 + 8x^2 & +32x & -48 \\ \hline \end{array}$$

Here, vertically multiplication,

Right side: $4 \times (-12) = - 48$ and

Left side: $(x^2) \times (x^2) = x^4$

Also crosswise multiplication,

For co-eff. of x: $48x + 4mx = 32x$ ∴ $4mx = (32 - 48)x$ ∴ $4mx = -16x$ ∴ $m = - 4$

For co-efficient of x^2 : $-12x^2 + 4x^2 - 4mx^2 = 8x^2$ ∴ $4mx^2 = -16x^2$ ∴ $m = - 4$

For co-efficient of x^3: $- 4x^3 + mx^3 = - 8x^3$ ∴ $mx^3 = -4x^3$ ∴ $m = -4$

∴Another quadratic expression must be $x^2 - 4x -12$ and

Factors of $x^2 - 4x - 12$ are $(x + 2)(x - 6)$.

Thus, $(x - 2), (x - 2), (x + 2)$ & $(x - 6)$ are factors.

5.3 Vertically & Crosswise Sūtra in solving Successive Differentiation:

5.3.1 Derivative of multiplication of 2 functions by Crosswise Sūtra:

Let u & w both be in the form of variable x, and if y = u. w then

$$\frac{dy}{dx} = u.\frac{dw}{dx} + w.\frac{du}{dx} \ldots\ldots\ldots [P]$$

Differentiation of multiplication of such type of a relationship is found out using Vedic Crosswise Sūtra, if one knows standard formula.

Example: 9

Find $\frac{dy}{dx}$ if $y = x^2.2^x$

By current method,

By using above formula [P],

$u = x^2$ & $w = 2^x$;

$$u' = \frac{du}{dx} = 2x \;\; \& \;\; w\,' = \frac{dw}{dx} = 2^x.\log 2$$

$$\therefore \frac{dy}{dx} = \;\; x^2\,(2^x \log 2) + 2^x.(2x)$$

$$\therefore \frac{dy}{dx} = x^2.2^x \log 2 + 2x.2^x$$

By Vedic Method,

Applying standard formula for finding derivative of u and w,

By using Vedic Crosswise Sūtra,

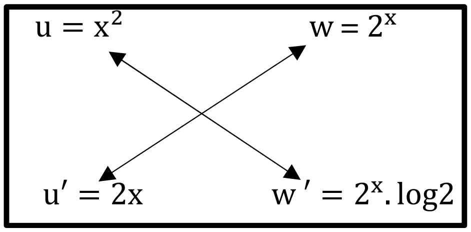


Figure: -5.1

From the above figure,

$$\therefore \frac{dy}{dx} = x^2(2^x \log 2) + 2x(2^x)$$

Example: 10

Find $\frac{dy}{dx}$ if $y = (2x^2 + 3x + 7)\,(5\,x^2 + 7)$

By using current method,

$u = (2x^2 + 3x + 7)$, $w = (5x^2 + 7)$

$$u\,' = \frac{du}{dx} = 4x + 3 \quad \& \;\; w\,' = \frac{dv}{dx} = 10x$$

$$\therefore \frac{dy}{dx} = (2x^2 + 3x + 7)10x + (5x^2 + 7)(4x + 3) \qquad \because \text{from}[P]$$

By using Vedic Method:

By using Dhvaja Ghata,

$u' = \frac{du}{dx} = \frac{d}{dx}(2x^2 + 3x + 7) \ \therefore u' = 4x + 3$

$\& \ w' = \frac{dw}{dx} = \frac{d}{dx}(5x^2 + 7) \quad \therefore w' = 10x$

By using Vedic **Crosswise Sūtra,**

$$\boxed{\begin{tikzcd} u = 2x^2 + 3x + 7 \arrow[dr, leftrightarrow] & w = 5x^2 + 7 \arrow[dl, leftrightarrow] \\ u' = 4x + 3 & w' = 10x \end{tikzcd}}$$

Figure: -5.2

$$\therefore \frac{dy}{dx} = (2x^2 + 3x + 7)10x + (5x^2 + 7)(4x + 3)$$

5.3.2 Procedure for finding 2nd order and 3rd order derivative by Crosswise Sūtra:

If f (x) is given. First find first order derivative i.e. $f'(x)$ by using Crosswise Sūtra.

By keeping first column, in the second column repeat by using the above Sūtra along with the binomial expansion (Pascal's Triangle) $f''(x)$ can be found.

For finding $f'''(x)$ repeat the above process which is equivalent to Leibnitz theorem.

By using binomial theorem and Vertically & Crosswise Sūtra,

Let f (x) = u. w

$$f'(x) = (u \, . \, w)' = 1C_0 \, u \, w' + 1C_1 \, u' \, w$$

$$f''(x) = (u \, .w)'' = 2C_0 \, u \, w'' + 2C_1 \, u' \, w' + 2C_2 \, u'' \, w$$

$$f'''(x) = (u. \, w)''' = 3C_0 u \, w''' + 3C_1 u' \, w'' + 3C_2 u'' \, w' + 3C_3 u''' \, w$$

$$f''''(x) = (u. \, w)'''' = 4C_0 \, u \, w'''' + 4C_1 \, u' \, w''' + 4C_2 \, u'' \, w'' + 4C_3 \, u''' \, w' + 4C_4 \, u'''' \, w$$

Example: 11

Find first and second order derivative for $y = (x^3 - 2)(x^2 + 3x + 2)$

Conventional Method:

Let $u = (x^3 - 2)$ and $w = (x^2 + 3x + 2)$

$\therefore u' = 3x^2$ & $w' = 2x + 3$

$\therefore \frac{dy}{dx} = \frac{d}{dx}(u.w) = (x^3 - 2)(2x + 3) + (x^2 + 3x + 2)(3x^2)$

By using Vedic Crosswise Sūtra,

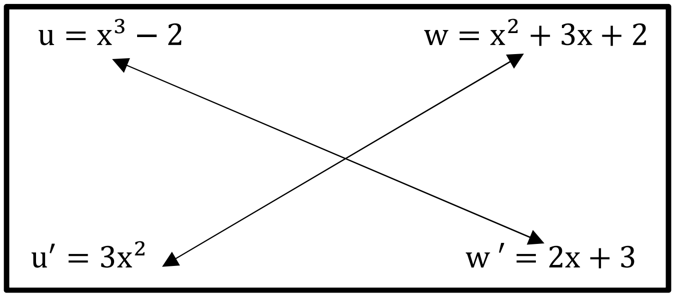


Figure: 5.3

$$\frac{dy}{dx} = (x^3 - 2)(2x + 3) + (x^2 + 3x + 2)(3x^2)$$

We can find 2nd order derivative from the 1st order derivative by Crosswise Sūtra.

=

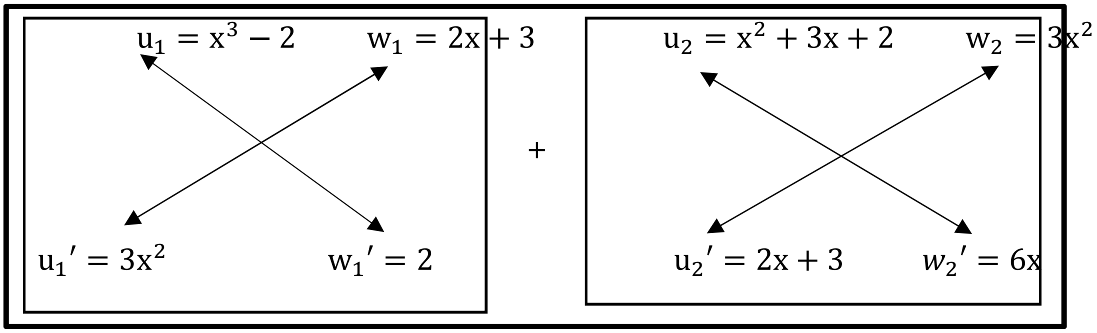


Figure: 5.4

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[(x^3 - 2)(2x + 3) + (x^2 + 3x + 2)(3x^2)]$$

$$= \frac{d}{dx}[(x^3 - 2)(2x + 3)] + \frac{d}{dx}[(x^2 + 3x + 2)(3x^2)]$$

$$= 2(x^3 - 2) + 3x^2(2x + 3) + (x^2 + 3x + 2)(6x) + 3x^2(2x + 3)$$

$$= 2(x^3 - 3) + 2[3x^2(2x + 3)] + (x^2 + 3x + 2)(6x)$$

Finding f ′′(x) by **Vertically & Crosswise Sūtra** and Binomial expansion:

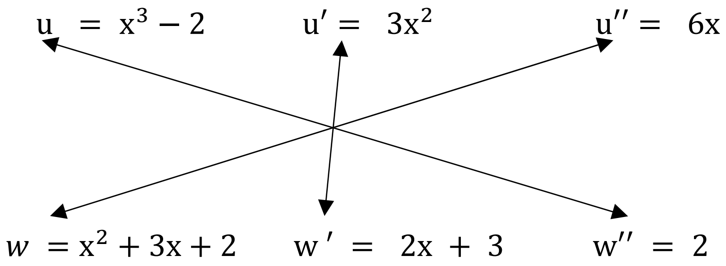


Figure: 5.5

$$\therefore \frac{d^2y}{dx^2} = 1[2(x^3 - 2)] + 2[3x^2(2x + 3)] + 1[(x^2 + 3x + 2)(6x)]$$

Example: 12

If $f(x) = y = x^4 . e^{3x}$, Find 2nd, 3rd and 4th order derivative of f(x).

Here, by using Vedic Vertically & Crosswise Sūtra,

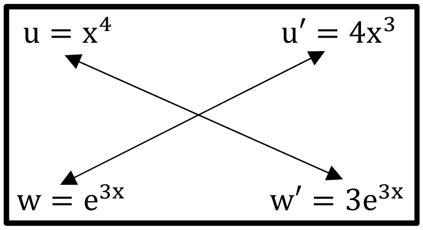


Figure: - 5.6

$$\therefore f'(x) = \frac{dy}{dx} = x^4(3e^{3x}) + e^{3x}(4x^3)$$

Finding $f''(x)$ by Vertically & Crosswise Sūtra and Binomial expansion:

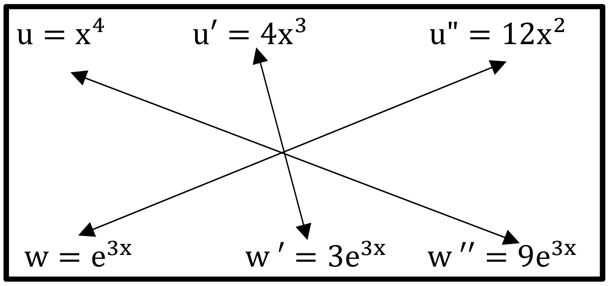


Figure: -5.7

Finding $f''(x)$ by

$f''(x) = 2C_0 u\, w'' + 2C_1 u'\, w' + 2C_2 u''\, w$

$$\therefore f''(x) = \frac{d^2y}{dx^2} = 1[x^4(9e^{3x})] + 2[3e^{3x}(4x^3)] + 1[e^{3x}(12x^2)]$$

Vertically & Crosswise Sūtra & Binomial expansion,

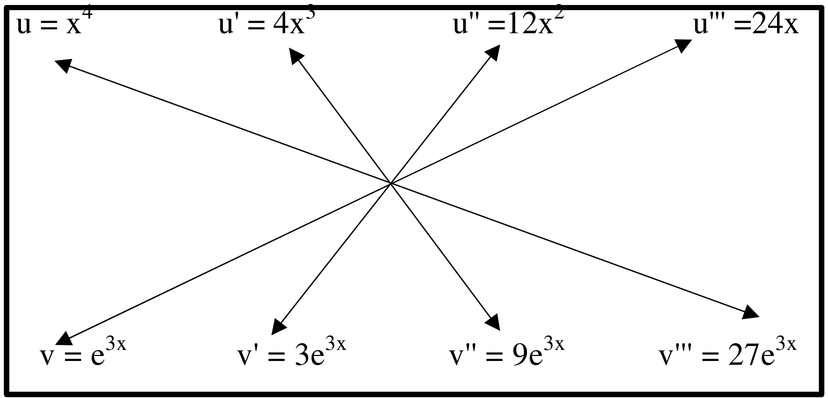


Figure: -5.8

Finding $f'''(x)$ by

$$f'''(x) = 3C_0uv''' + 3C_1u'v'' + 3C_2u''v'' + 3C_3u'''v$$

$$\therefore f'''(x) = \frac{d^3y}{dx^3} = 1[x^4(27e^{3x})] + 3[9e^{3x}(4x^3)] + 3[3e^{3x}(12x^2)] + 1[e^{3x}(24x)]$$

$$= 27x^4e^{3x} + 108\,x^3e^{3x} + 108x^2e^{3x} + 24xe^{3x}$$

By using V & C Sūtra along with binomial expansion is explained in figure below Finding $f''''(x)$:

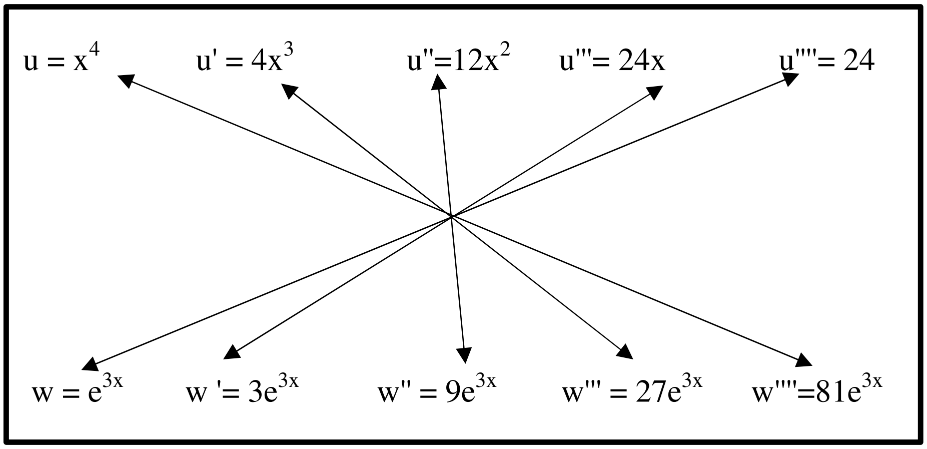


Figure: - 5.9

$$f''''(x) = 4c_0u\,w'''' + 4c_1u'w''' + 4c_2u''w'' + 4c_3u'''w' + 4c_4u''''w$$

$$\therefore\ f''''(x) = 1[(x^4)\,(81e^{3x})] + 4[(4x^3)\,(27e^{3x})] + 6\,[(12x^2)\,(9e^{3x})]$$

$$+ 4 [(3e^{3x}) (24x)] + 1[24 e^{3x}]$$

$$= 81 x^4e^{3x} + 432 x^3e^{3x} + 648 x^2e^{3x} + 288x e^{3x} +24 e^{3x}$$

$$= e^{3x} [81x^4 + 432x^3 +648x^2 +288x +24]$$

5.3.3 Procedure for finding derivative of product of three functions:

If we want to find out the above value, then consider the same procedure as for finding derivative of two functions by considering u .w = z.

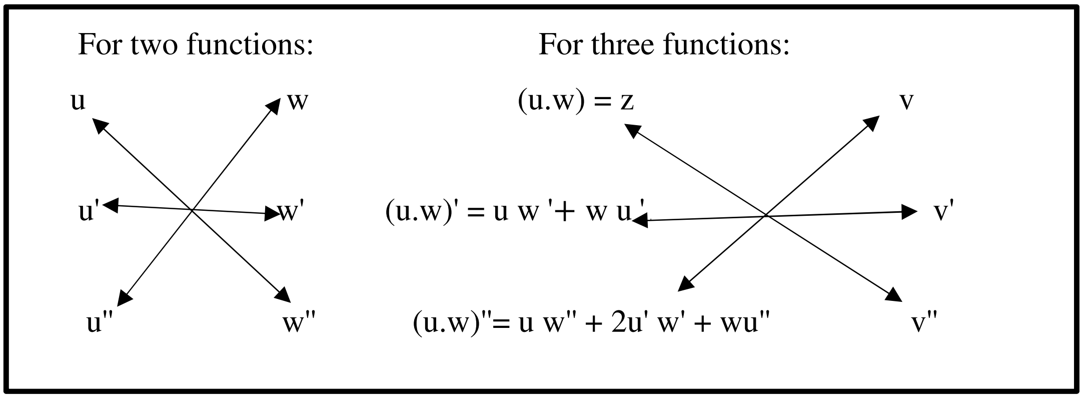


Figure: 5.10

If z = f (x) = u .w then f ''(x) = u w '' + 2 u' w ' + w u '' and

If y = f(x) = u. w. v then

f ''(x) = z v ''+ 2 (u . w '+ w u ') v ' + (u w''+ 2 u'w'+ wu'') v

= (u. w) v '' + 2 (u. w) 'v ' + (u. w)'' v

Example: 13

If $y = e^x.x^4.\log x$ then find f ''(x).

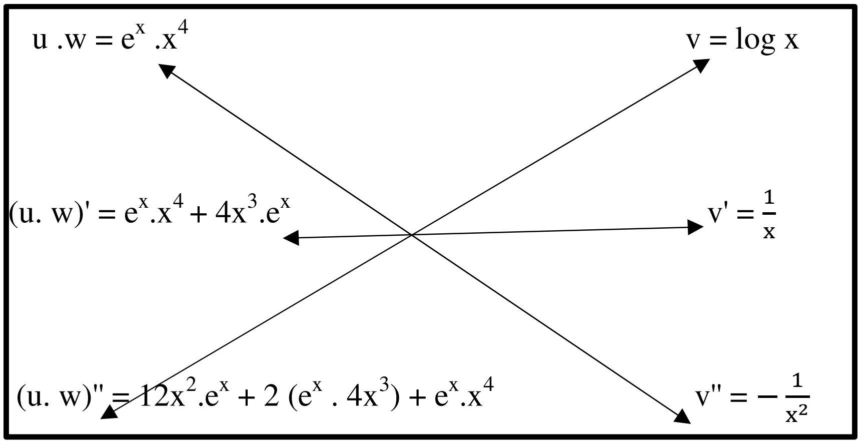

$u .w = e^x .x^4$
$v = \log x$
$(u. w)' = e^x.x^4 + 4x^3.e^x$
$v' = \frac{1}{x}$
$(u. w)'' = 12x^2.e^x + 2 (e^x . 4x^3) + e^x.x^4$
$v'' = -\frac{1}{x^2}$

Figure: 5.11

$f''(x) = (u.\ w)\ v'' + 2\ (u.\ w)'\ v' + (u.\ w)''\ v$

$$= (e^x.x^4)\left[-\frac{1}{x^2}\right] + 2\ (e^x.\ x^4 + 4x^3.e^x)\left[\frac{1}{x}\right] + [12x^2.\ e^x + 2\ (e^x.4x^3) + e^x.x^4]\log x$$

$$= e^x\,[-x^2 + 2x^3 + 8x^2 + 12x^2 \log x + 8x^3 \log x + x^4 \log x]$$

$$\therefore f''(x) = e^x\,[(2x^3 + 6x^2) + (12x^2 + 8x^3 + x^4)\log x]$$

Example: 14

Find $\frac{d^2y}{dx^2}$ if $y = \frac{\log x}{e^x}$

By using conventional method by using division rule,

$$\frac{dy}{dx} = \frac{e^x.\frac{d}{dx}(\log x) - \log x.\ \frac{d}{dx}(e^x)}{(e^x)^2}$$

$$= \frac{e^x.\ \frac{1}{x} - \log x.e^x}{(e^x)^2}$$

$$= e^{-x}\frac{1}{x} - e^{-x}\log x$$

By using Crosswise Sūtra:

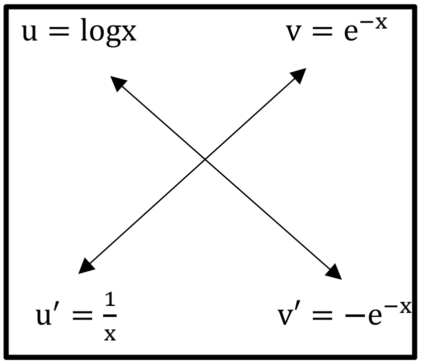


Figure: -5.12

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log x}{e^x}\right) = \frac{d}{dx}(\log x.e^{-x}) = e^{-x}.\frac{1}{x} + \log x(-e^{-x})$$

$$= e^{-x}.\frac{1}{x} - e^{-x}.\log x$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^{-x}.\quad \frac{1}{x} - e^{-x}.\log x\right]$$

$$= \frac{d}{dx}[e^{-x}(\frac{1}{x} - \log x)]$$

Again applying product rule by **Crosswise Sūtra:**

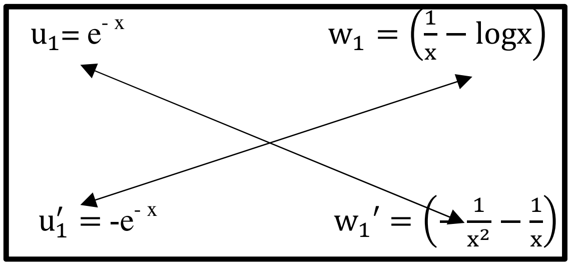

$u_1 = e^{-x}$
$w_1 = \left(\frac{1}{x} - \log x\right)$
$u_1' = -e^{-x}$
$w_1' = \left(-\frac{1}{x^2} - \frac{1}{x}\right)$

Figure: - 5. 13

$$\therefore \frac{d^2y}{dx^2} = e^{-x}.\left(-\frac{1}{x^2} - \frac{1}{x}\right) + \left(\frac{1}{x} - \log x\right)(-e^{-x})$$

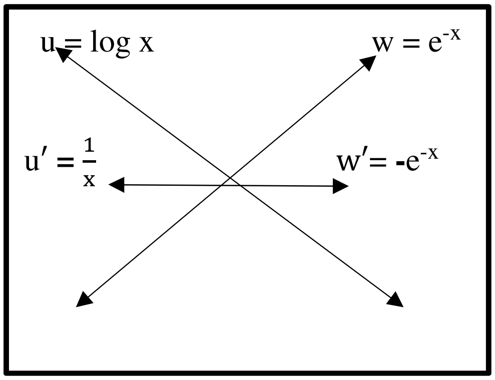
$$= e^{-x}\left[-\frac{1}{x^2} - \frac{1}{x} - \frac{1}{x} + \log x\right]$$

$$\therefore \frac{d^2y}{dx^2} = e^{-x}\left[\log x - 2\left(\frac{1}{x}\right) - \left(\frac{1}{x^2}\right)\right]$$

5.3.4 Alternative way to find out Second order derivative:

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}\left(\frac{\log x}{e^x}\right) = \frac{d^2}{dx^2}(\log x.\, e^{-x})$$

By using **Vertically & Crosswise Sūtra**,



$u'' = -\frac{1}{x^2}$ $w'' = e^{-x}$

Figure: - 5.14(a)

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(\log x . e^{-x}) = 1[e^{-x}\log x] + 2\left[e^{-x}.\frac{1}{x}\right] + 1\left[e^{-x}.\left(-\frac{1}{x^2}\right)\right]$$

$$\therefore \frac{d^2y}{dx^2} = e^{-x}\left[\log x - \frac{2}{x} - \frac{1}{x^2}\right]$$

5.4 Digression: Differentiation of the ratio of the polynomials:

Derivative of the division of two polynomials:

If u and w both are polynomials then by using Vertically & Crosswise Sūtra derivative of division of two polynomial functions can be easily find out.

Example: 15

Differentiate:

$$y = \frac{2+4x}{2x+2x^2}$$

By applying division rule,

$$\frac{dy}{dx} = \frac{(2x+2x^2)4 - (2+4x)(2+4x)}{[2x+2x^2]^2}$$

$$= \frac{(8x+8x^2) - (4+16x+16x^2)}{[2x+2x^2]^2}$$

$$= \frac{8x+8x^2-4-16x-16x^2}{[2x+2x^2]^2}$$

$$\therefore \frac{dy}{dx} = \frac{-4-8x-8x^2}{[2x+2x^2]^2}$$

The above method is very lengthy, for finding derivative of division of two polynomial by using crosswise sūtra the numerator part of the above answer can be easily obtained which is explained in the figure as below and denominator is the square of the term which is in the denominator of the given problem.

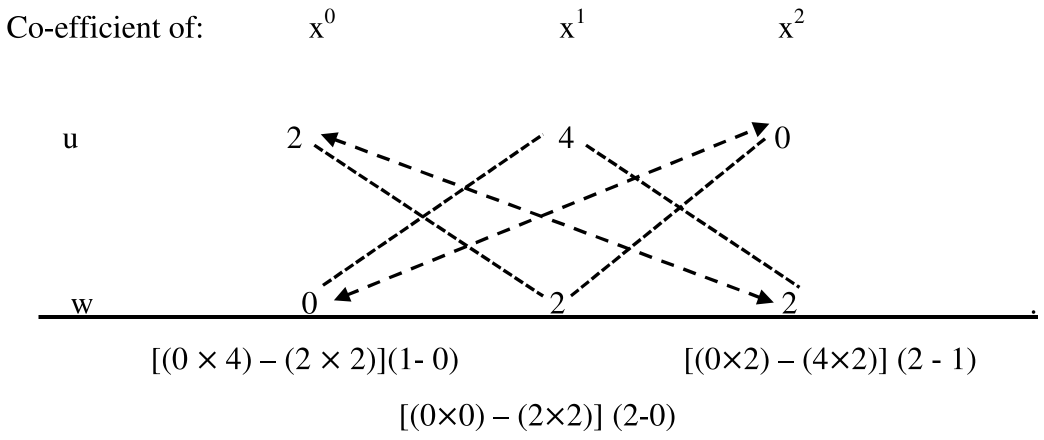


Figure: 5.14(b)

Let $y = \frac{2 + 4x}{2x + 2x^2} = \frac{2x^0 + 4x + 0x^2}{0x^0 + 2x + 2x^2}$

$$\frac{dy}{dx} = \frac{(0 \times 4 - 2 \times 2)(1 - 0) + (0 - 2 \times 2)(2 - 0)x + (2 \times 0 - 4 \times 2)(2 - 1)x^2}{[2x + 2x^2]^2}$$

$$= \frac{(-4) + (-4)2x + (-8)x^2}{[2x + 2x^2]^2}$$

$$\therefore \frac{dy}{dx} = \frac{-4 - 8x - 8x^2}{[2x + 2x^2]^2}$$

Example: 16

Differentiate:

$$y = \frac{1 + 4x + 9x^2 + 3x^3}{2 + 3x + 4x^2 + 9x^3}$$

According to current method by using division rule,

$$\frac{dy}{dx} = \frac{(2 + 3x + 4x^2 + 9x^3)(4 + 18x + 27x^2) - (1 + 4x + 9x^2 + 27x^2)(3 + 8x + 27x^2)}{[(2 + 3x + 4x^2 + 9x^3)]^2}$$

$$\therefore \frac{dy}{dx} = \frac{5 + 28x + 2x^2 - 54x^3 - 69x^4}{[2 + 3x + 4x^2 + 9x^3]^2}$$

By using V & C Sūtra,

$$\therefore \frac{dy}{dx} = \frac{\begin{gathered}(2\times 4-1\times 3)(1-0)+(2\times 9-1\times 4)(2-0)x+(2\times 3-1\times 9)(3-0)x^2+\\(3\times 9-4\times 4)(2-1)x^2+(3\times 3-4\times 9)(3-1)x^3+(3\times 4-9\times 9)(3-2)x^4\end{gathered}}{[2+3x+4x^2+9x^3]^2}$$

$$\therefore \frac{dy}{dx} = \frac{\begin{gathered}(8-3)(1)+(18-4)(2)x+(6-9)(3)x^2+(27-16)(1)x^2\\+(9-36)(2)x^3+(12-81)(1)x^4\end{gathered}}{[2+3x+4x^2+9x^3]^2}$$

$$\therefore \frac{dy}{dx} = \frac{5+28x-9x^2+11x^2-54x^3-69x^4}{[2+3x+4x^2+9x^3]^2}$$

$$\therefore \frac{dy}{dx} = \frac{5+28x+2x^2-54x^3-69x^4}{[2+3x+4x^2+9x^3]^2}$$

5.5 Leibniz Rule for finding n^{th} order derivative of multiplication of 2 Functions:

Let u & w be the functions of x and their derivatives up to n^{th} order exists then:

$$f^{(n)}(x)\boxed{\begin{aligned}&= (uw)^n = nC_0uw^{(n)} + nC_1u'\,w^{(n-1)} + nC_2u''w^{(n-2)} + nC_3u'''w^{(n-3)}\\&\qquad + \;.........nC_2u^{(n-2)}w'' + nC_{n-1}u^{(n-1)}w' + u^{(n)}w\;......[A]\end{aligned}}$$

$$\text{i.e.}\ (uw)^n = \sum_{i=0}^{n}\binom{n}{i}\,u^{(i)}w^{(n-i)}$$

Where $\binom{n}{i}$ denote the number of I - combinations of n elements.

Example: 17

Find n^{th} order derivative of $y = x^4.\,e^{3x}$

Let $u = x^4;\ u' = 4x^3; u'' = 12x^2; u''' = 24x; u^{(iv)} = 24;\ u^{(v)} = 0$

And $w = e^{3x}$; $w' = e^{3x}$; $w'' = e^{3x}$; .... $w^{(n)} = e^{3x}$

By using Leibniz rule in the above [A],

Here, $u^{(v)} = 0$

∴ It can be applied till 4th power of u.

$\therefore f^{(n)}(x) = (uw)^n$

$$= nC_0uw^{(n)} + nC_1u'\,w^{(n-1)} + nC_2u''w^{(n-2)} + nC_3u'''w^{(n-3)} + nC_4u^{(iv)}w^{(n-4)}$$

$$= x^4e^{3x} + 4nx^3e^{3x} + \frac{12}{2!}n(n-1)x^2e^{3x} + \frac{24}{3!}n(n-1)(n-2)xe^{3x} +$$

$$\frac{24}{4!}n(n-1)(n-2)(n-3)e^{3x} + 0 + \ldots.0 + 0.$$

$$= x^4e^{3x} + 4nx^3e^{3x} + 6n(n-1)x^2e^{3x} + 4n(n-1)(n-2)x\,e^{3x}$$

$$+ n(n-1)(n-2)(n-3)e^{3x} + \cdots$$

$\therefore f^{(n)}(x)$

$= e^{3x}[x^4 + 4nx^3 + 6n(n-1)x^2 + 4n(n-1)(n-2)x + n(n-1)(n-2)(n-3)]$

By Vertically & Crosswise Sutra and Binomial expansion for nth order,

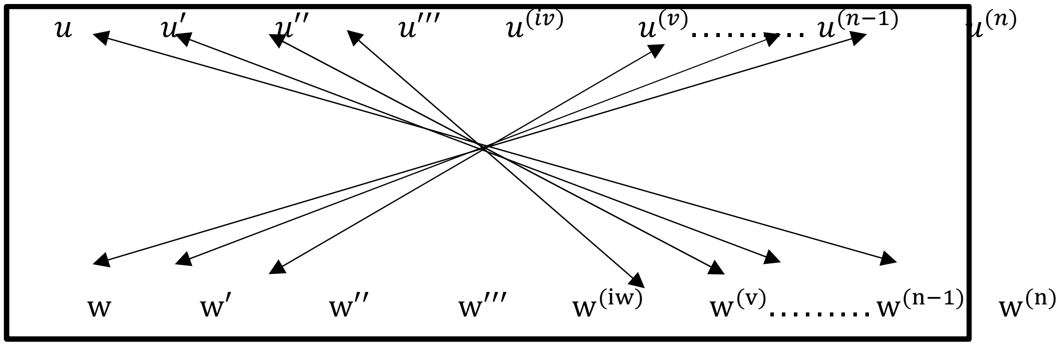


Figure: 5.15

$$f^{(n)}(x) = nC_0uw^{(n)} + nC_1u'\,w^{(n-1)} + nC_2u''w^{(n-2)} + nC_3u'''w^{(n-3)} + \ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots + nC_{n-2}u^{(n-2)}w'' + nC_{n-1}u^{(n-1)}w' + nC_nu^{(n)}w \ldots\ldots\ldots [B]$$

Which is equivalent to equation [A] above.

So, we get the same answer but in one step from the above figure by using Vedic method along with binomial expansion.

5.6 Integration by using Vedic Sūtras:

5.6.1 Integration by using Ekādhikena Pūrveṇa Sūtra:

Ekādhikena Pūrveṇa Sūtra is used for finding the integral of the type x^n.
According to Sūtra add one to the given power of x and divide the obtained term by the new power of x. i.e. by the given power plus one.

[1] $\int 28x^3 \; dx = \frac{28}{4}x^4 + c.$

[2] $\int (x^4 + 2x^3 + 9x^2 + 6x - 2) \; dx = \frac{1}{5}x^5 + \frac{2}{4}x^4 + \frac{9}{3}x^3 + \frac{6}{2}x^2 - 2x + c.$

5.7 Integration based on Partial fraction by Parāvartya Yojayet Sūtra:

If f(x) and h(x) are two polynomials, then $\frac{f(x)}{h(x)}$ is a new relationship in terms of x whose rationality is dependent on the range of values its denominator can take i.e. if h(x) has non-zero value, the function comes under rational category and if further its degree is greater than the degree of f(x), then $\frac{f(x)}{h(x)}$ is of proper type. It can be expressed by partial fractions using following table in which L, M, N and O are real numbers.

Rational Form	Partial Form
$\frac{Px^2 + qx + r}{(x-a)(x-b)(x-c)}$	$\frac{L}{(x-a)} + \frac{M}{(x-b)} + \frac{N}{(x-c)}$
$\frac{Px^2 + qx + r}{(x-a)^2(x-b)}$	$\frac{L}{(x-a)} + \frac{M}{(x-a)^2} + \frac{N}{(x-b)}$
$\frac{Px^2 + qx + r}{(x-a)^3(x-b)}$	$\frac{L}{(x-a)} + \frac{M}{(x-a)^2} + \frac{N}{(x-a)^3} + \frac{O}{(x-b)}$
$\frac{Px^2 + qx + r}{(x-a)(x^2+bx+c)}$	$\frac{L}{(x-a)} + \frac{Mx+N}{x^2+bx+c}$ Where $x^2 + bx + c$ can't be factorized Integration Partial fraction is an important.

	Solving the problem of integration on partial fraction by using Parāvartya Yojayet Sūtra is easily than the current laborious method.further.

Table - 5.1

Partial fractions means decomposing the denominator into irreducible factors.

5.7.1 Denominator containing linear and unrepeated factors:

Example: 18

Evaluate:

$$\int \frac{3x^2 - x - 4}{(x-1)(x-2)(x-3)}\ dx$$

First express the above in the form of partial fraction.

By using current method:

Let

$$E = \frac{3x^2 - x - 4}{(x-1)(x-2)(x-3)} = \frac{L(x^2 - 5x + 6) + M(x^2 - 4x + 3) + N(x^2 - 3x + 2)}{(x-1)(x-2)(x-3)}$$

$$\therefore 3x^2 - x - 4 = L(x^2 - 5x + 6) + M(x^2 - 4x + 3) + N(x^2 - 3x + 2)$$

$$= x^2(L + M + N) - x(5L + 4M + 3N) + (6L + 3M + 2N)$$

Equating the co-efficient of same exponenta on L.H.S. as well as R.H.S.**,** we get

$$L + M + N = 3 \ldots\ldots\ldots\ldots\ldots(i)$$

$$5L + 4M + 3N = 1 \ldots\ldots\ldots\ (ii)$$

$$6L + 3M + 2N = -4 \ldots\ldots.\ (iii)$$

Solving equations (i), (ii) & (iii)

$5(3 - M - N) + 4M + 3N = 1 \ \therefore\ 15 - 5M - 5N + 4M + 3N = 1$

$\therefore -M - 2N = 14 \ \therefore\ M + 2N = 14 \ldots\ldots\ldots\ (iv)$

$6(3 - M - N) + 3M + 2N = -4 \ \therefore 18 - 6M - 6N + 3M + 2N = -4$

$\therefore\ -3M - 4N = 22 \ \therefore 3M + 4N = 22 \ldots\ (v)$

Solving Equations (iv) Lnd (v),

By solving, M + 2N = 14 and 3M + 4N = 22 we get L = -1, M = - 6 Lnd N = 10

By using Vedic Method:

An alternate method for solving problems based on Partial fractions which is easier & faster is by employing Parāvartya Yojayet Sūtra.This method helps in combating the drawbacks of extensive computations and extra efforts of simultaneous equations method. Let

$$\frac{3x^2-x-4}{(x-1)(x-2)(x-3)}=\frac{L}{x-1}+\frac{M}{x-2}+\frac{N}{x-3}\dots\dots\dots.[I]$$

For finding value of L:

By considering x- 1= 0, we get x = 1.

By substituting Parāvartya value x = 1 in equation [1] neglecting the x-1 term in the denominator.

Thus, we get the value of L as follows:

$$L=\frac{3x^2-x-4}{(x-2)(x-3)}=\frac{3(1)^2-1-4}{(1-2)(1-3)}=\frac{3-5}{(-1)(-2)}=\frac{-2}{2}=-1$$

Similarly, for finding value of M put x = 2

$$M=\frac{3x^2-x-4}{(x-1)\,(x-3)}=\frac{3(2)^2-2-4}{(2-1)\,(2-3)}=\frac{12-6}{(1)\,(-1)}=\frac{6}{-1}=-6$$

And for finding value of N value put x = 3

$$N=\frac{3x^2-x-4}{(x-1)\,(x-2)}=\frac{3(3)^2-3-4}{(3-1)\,(3-2)}=\frac{27-7}{(2)(1)}=\frac{20}{2}=10$$

Thus, using Vedic Method, L, M and N can be calculated mentally as:

L = -1, M = - 6 and N = 10

$$\int\frac{3x^2-x-4}{(x-1)(x-2)(x-3)}\,dx=\int\frac{-1}{x-1}\,dx+\int\frac{-6}{x-2}\,dx+\int\frac{10}{x-3}\,dx$$

$$\therefore\int\frac{3x^2-x-4}{(x-1)(x-2)(x-3)}\,dx$$

$$=(-1)\log|x-1|-6\log|x-2|+10\log|x-3|+c$$

5.7.2. Denominator containing linear and repeated factors:

Example: 19

Evaluate:

$$\int \frac{3x^2 + 12x + 11}{(x-1)(x-2)^2}\ dx$$

First express the above in Partial fraction.

By using Current method:

$$\text{Let E} = \frac{3x^2 + 12x + 11}{(x-1)(x-2)^2} = \frac{L}{x-1} + \frac{M}{x-2} + \frac{N}{(x-2)^2} \ldots\ldots.[A]$$

$$\therefore \frac{3x^2 + 12x + 11}{(x-1)(x-2)^2} = \frac{L(x^2 - 4x + 4) + M(x^2 - 3x + 2) + N(x-1)}{(x-1)(x-2)^2}$$

$$\therefore 3x^2 + 12x + 11 = L(x^2 - 4x + 4) + M(x^2 - 3x + 2) + N(x-1)$$
$$= x^2(L + M) + x(-4L - 3M + N) + (4L + 2M - N)$$

Equating the Co-efficient of like powers on both sides, we get

$L + M = 3$…………………….. (i)

$(-4L - 3M + N) = 12$………….. (ii)

$(4L + 2M - N) = 11$…………… (iii)

∴ By solving equations (i), (ii) & (iii)

$-4(3 - M) - 3M + N = 12$

$\therefore -12 + 4M - 3M + N = 12 \therefore M + N = 24$……… (iv)

$4(3 - M) + 2M - N = 11 \therefore 12 - 4M + 2M - N = 11 \therefore -2M - N = -1$... (v)

Solving Equations (iv) and (v),

By solving, $M + N = 24$ and $-2M - N = -1$ we get $L = 26$, $M = -23$ and $N = 47$.

By using Vedic Method,

Let,

$$\frac{3x^2 + 12x + 11}{(x-1)(x-2)^2} = \frac{L}{x-1} + \frac{M}{x-2} + \frac{N}{(x-2)^2} \ldots [I]$$

For finding value of L:

By putting $x - 1 = 0$ and thus we have $x = 1$ as a Parāvrtya Value.

Thus, we get the value of L as follows:

$$\mathrm{L} = \frac{3\mathrm{x}^2 + 12\mathrm{x} + 11}{(\mathrm{x}-1)(\mathrm{x}-2)^2} = \frac{3(1)^2 + 12 + 11}{(1-2)^2} = \frac{26}{(-1)^2} = \frac{26}{1} = 26$$

Similarly for finding value of N put x = 2

$$\mathrm{N} = \frac{3\mathrm{x}^2 + 12\mathrm{x} + 11}{(\mathrm{x}-1)(\mathrm{x}-2)^2} = \frac{3(2)^2 + 12(2) + 11}{(2-1)} = \frac{12 + 24 + 11}{(1)} = \frac{47}{1} = 47$$

To find the value of M put x = 0 in equation [I]

$$\therefore \frac{3(0)^2 + 12(0) + 11}{(0-1)(0-2)^2} = \frac{\mathrm{L}}{(0-1)} + \frac{\mathrm{M}}{(0-2)} + \frac{\mathrm{N}}{(0-2)^2}$$

$$\therefore \frac{11}{(-1)(-2)^2} = \frac{\mathrm{L}}{(-1)} + \frac{\mathrm{M}}{(-2)} + \frac{\mathrm{N}}{(-2)^2}$$

$$\therefore \frac{11}{-4} = \frac{\mathrm{L}}{-1} + \frac{\mathrm{M}}{-2} + \frac{\mathrm{N}}{-4}$$

$$\therefore -11 = -4\mathrm{L} - 2\mathrm{M} + \mathrm{N}$$

$$\therefore 11 = 4\mathrm{L} + 2\mathrm{M} - \mathrm{N}$$

By substituting value of L = 26 and N = 47 we get,

4(26) + 2M- 47=11 $\therefore 104 + 2\mathrm{M} - 47 = 11$

$$\therefore 2\mathrm{M} = 11 + 47 - 104 = -46 \therefore \mathrm{M} = \frac{-46}{2} = -23$$

Thus, using Vedic Method, the value of L, M and N can be found out mentally.

L = 26, M = - 23 and N = 47 substitute in equation [I],

$$\frac{3\mathrm{x}^2 + 12\mathrm{x} + 11}{(\mathrm{x}-1)(\mathrm{x}-2)^2} = \frac{26}{\mathrm{x}-1} + \frac{-23}{\mathrm{x}-2} + \frac{47}{(\mathrm{x}-2)^2}$$

$$\int \frac{3\mathrm{x}^2 + 12\mathrm{x} + 11}{(\mathrm{x}-1)(\mathrm{x}-2)^2}\,\mathrm{dx} = \int \frac{26}{\mathrm{x}-1}\,\mathrm{dx} + \int \frac{-23}{\mathrm{x}-2}\,\mathrm{dx} + \int \frac{47}{(\mathrm{x}-2)^2}\,\mathrm{dx}$$

$$= 26\int \frac{1}{\mathrm{x}-1}\mathrm{dx} - 23\int \frac{1}{\mathrm{x}-2}\,\mathrm{dx} + 47\int \frac{1}{(\mathrm{x}-2)^2}\,\mathrm{dx}$$

$$= 26 \log|x-1| - 23 \log|x-2| + 47\left(-\frac{1}{x-2}\right) + c$$

$$\therefore \int \frac{3x^2+12x+11}{(x-1)(x-2)^2}\,dx = 26 \log|x-1| - 23 \log|x-2| - 47\left(\frac{1}{x-2}\right) + c$$

5.7.3 Denominator containing the quadratic equation which cannot be factorized

Example: 20

Evaluate:

$$\int \frac{8}{(x+2)(x^2+4)}\,dx$$

By current method,

Let

$$\frac{8}{(x+2)(x^2+4)} = \frac{L}{x+2} + \frac{Mx+N}{x^2+4}$$

Then , $8 = L(x^2+4) + (Mx+N)(x+2)$……… (i)

Putting x = -2 in equation (i), $8 = 8L \Rightarrow \boxed{L = 1}$

Comparing coefficient of x^2, x and constant number on both sides of (i)

$$8 = Lx^2 + 4L + Mx^2 + Nx + 2Mx + 2N$$

$$= x^2(L+M) + x(N+2M) + 4L + 2N$$

Comparing coefficient of x^2

$$\Rightarrow L + M = 0 \Rightarrow L = -M \Rightarrow \boxed{M = -1}$$

Co-efficient of x $\Rightarrow 2M + N = 0 \Rightarrow 2(-1) + N = 0 \Rightarrow \boxed{N = 2}$

Compare constant form

$$4L + 2N = 8 \Rightarrow 4(1) + 2(2) = 8$$

By using Vedic method:

Let

$$\frac{8}{(x+2)(x^2+4)} = \frac{L}{x+2} + \frac{Mx+N}{x^2+4} \ldots\ldots..[I]$$

In order to calculate A, by considering x + 2 = 0 and thus we get **x =** - 2 as Parāvartya value and by putting it in equation [1] in which the term in the denominator of A is neglected.

Thus, we get the value of L as follows:

$$L = \frac{8}{(x + 2)(x^2 + 4)} = \frac{8}{(-2)^2 + 4} = \frac{8}{8} = 1 \therefore \boxed{L = 1}$$

Similarly for finding value of N put x = 0 in [I]

$$\therefore \frac{8}{(0 + 2)(0 + 4)} = \frac{L}{0 + 2} + \frac{M(0) + N}{0 + 4}$$

$$\therefore \frac{8}{8} = \frac{L}{2} + \frac{N}{4} \therefore 1 = \frac{2L + N}{4}$$

But $L = 1 \therefore 4 = 2 + N \therefore \boxed{\therefore N = 2}$

To find the value of M put x = - 1 in equation [I]

$$\therefore \frac{8}{[(-1) + 2][(-1)^2 + 4]} = \frac{L}{(-1) + 2} + \frac{M(-1) + N}{(-1)^2 + 4}$$

Also substitute L = 1 & N = 2

$$\therefore \frac{8}{5} = 1 + \frac{2 - M}{5} \quad \therefore 8 = 5 + 2 - M$$

$$\therefore M = 7 - 8 \boxed{\therefore M = -1}$$

Thus, the value of L = 1, M = -1 and N = 2 substitute in Equation [I]

$$\int \frac{8}{(x + 2)(x^2 + 4)} dx = \int \frac{1}{x + 2} dx + \int \frac{-x + 2}{x^2 + 4} dx$$

$$= \int \frac{1}{x + 2} dx - \int \frac{x}{x^2 + 4} dx + 2\int \frac{1}{x^2 + 4} dx$$

$$= \log|x + 2| - \frac{1}{2}\log|x^2 + 4| + 2.\frac{1}{2}\tan^{-1} + c$$

$$\therefore \int \frac{8}{(x+2)(x^2+4)}\,dx = \log|x+2| - \frac{1}{2}\log|x^2+4| + \tan^{-1}\left(\frac{x}{2}\right) + c$$

$$\left[\because \text{ using } \int \frac{f\prime(x)}{f(x)}\,dx = \log\ |f(x)|\right]\ \&\ \left[\text{ using } \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

5.8 Integration by parts:

The method of integration by parts is used when the integrand is expressed as a product of two functions, one of which can be differentiated (u) and the other can be integrated (w) conveniently. Let w and u are expressed in terms of variable x.

$$\boxed{\int u.w\,dx = u\int w dx - \int \left\{\frac{d}{dx}(u)\int w\,dx\right\}dx}$$

When integrand is a product of two functions, out of which the first has to be differentiate and the second has to be integrated (whose integral is known), hence we should make the proper choice of first and second function.

We can also choose the first function as the function which comes first in the word '**LIATE'**
where L-logarithmic function I for Inverse, A-algebraic T-trigonometry
And E-exponential

5.8.1 Integration of the product of two functions by Vertically and Crosswise :

For the integration of the product of two functions (of x), can be solve by the following formula by multiplying it crosswise and add it with alternative sign.

$$\boxed{\int u.w\,dx = uw' - u_1w'' + u_2w''' - u_3w'''' + u_4w''''' - \ldots\ldots\ldots\ldots}\ldots\ldots\ldots\ldots [B]$$

Example: 21

Find the following integrals:

[1]$\int x^2e^{2x}dx$

According to current method,

Let I $= \int x^2e^{2x}dx$

$u = x^2$ and $w = e^{2x}$ & by using integration by parts formula,

$$I = x^2 \int e^{2x}\ dx - \int \left(\frac{d}{dx}\ (x^2)\ \int e^{2x}\ dx\right)\ dx$$

$$= x^2 .\ \frac{e^{2x}}{2} - \int \left[(2x)\ \left(\frac{e^{2x}}{2}\right)\right]\ dx = \frac{x^2 e^{2x}}{2} - \int x\, e^{2x}\ dx$$

Again by using integration by parts method for $\int x\ e^{2x}\ dx$

$$= \frac{1}{2}\, x^2 e^{2x} - \left[x \int e^{2x} dx - \int \left(\frac{d}{dx}(x) \int e^{2x} dx\right) dx\right]$$

$$= \frac{1}{2}\, x^2 e^{2x} \left[x \int e^{2x} dx - \int \left[(1)\left(\frac{e^{2x}}{2}\right)\right]\ dx\right]$$

$$= \frac{1}{2}\, x^2 e^{2x} - \left[\frac{1}{2}\ x\, e^{2x} - \frac{1}{2} \int e^{2x}\, dx\right]$$

$$= \frac{1}{2}\, x^2 e^{2x} - \left[\frac{1}{2}\ x\, e^{2x} - \frac{1}{2}\ \left[\frac{e^{2x}}{2}\right]\right] + c$$

$$= \frac{1}{2}\, x^2 e^{2x} - \left[\frac{1}{2}\ x\, e^{2x} - \frac{1}{4}\ e^{2x}\right] + c$$

$$\therefore I = \frac{1}{2}\, x^2 e^{2x} - \frac{1}{2}\ x\, e^{2x} + \frac{1}{4}\ e^{2x} + c$$

By using Vertically & Crosswise Sūtra:

Let $I = \int x^2 e^{2x} dx$

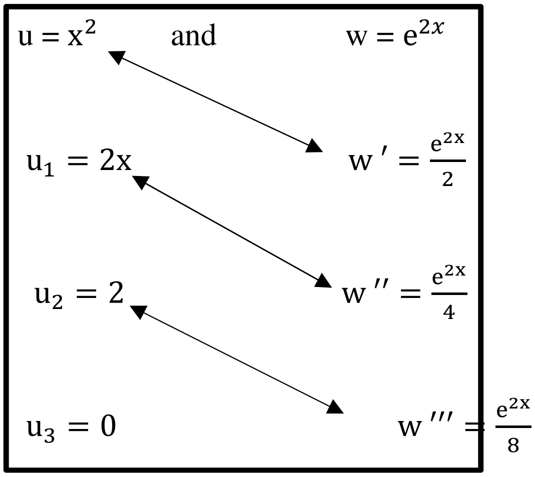


Figure- 5.16

By using equation [B]

$$\therefore \int x^2 e^{2x} dx = \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x\, e^{2x} + \frac{1}{4} e^{2x} + c$$

[2] $\int x \,.\, 2^{-3x}\, dx$

Let $I = \int x \,.\, 2^{-3x}\, dx$. Taking $u = x$ & $w = 2^{-3x}$ by using integration by parts

$$I = x \int 2^{-3x}\, dx - \int \left[\frac{d}{dx}(x) \int 2^{-3x}\, dx\right] dx$$

$$= x \,.\, \frac{2^{-3x}}{-3\,(\log 2)} - \int 1 \,.\, \left[\frac{2^{-3x}}{-3\, \log\, 2}\right] dx$$

$$I = \frac{x \,.\, 2^{-3x}}{3\, \log\, 2} + \frac{1}{3 \log 2} \int 2^{-3x}\, dx$$

$$= -\frac{1}{3 \log 2}\, x 2^{-3x} + \frac{1}{3 \log 2} \left[\frac{2^{-3x}}{(-3)\, \log 2}\right] + c$$

$$\therefore I = -\frac{1}{3 \log 2}\, x.\, 2^{-3x} - \frac{2^{-3x}}{9\,(\log 2)^{2}} + c$$

By Vertically and Crosswise Sūtra :

Let $I = \int x \,.\, 2^{-3x}\, dx$

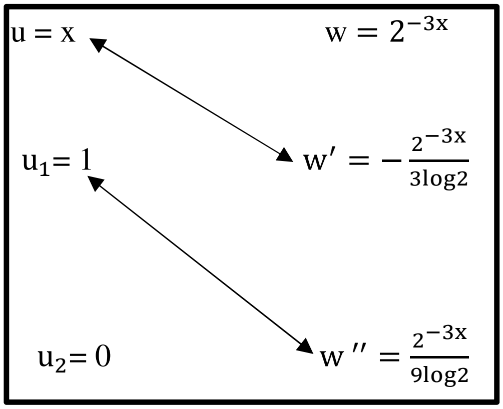

$u = x$
$w = 2^{-3x}$
$u_1 = 1$
$w' = -\frac{2^{-3x}}{3\log 2}$
$u_2 = 0$
$w'' = \frac{2^{-3x}}{9\log 2}$

Figure: - 5.17

$$\int x \,.\, 2^{-3x}\, dx = uw' - u_1 w'' + u_2 w'''$$

$$\therefore \int x.\, 2^{-3x}\, dx = -\frac{1}{3 \log 2}\, x\, .2^{-3x} - \frac{2^{-3x}}{9\,(\log 2)^{2}} + c$$

[3] $\int \log x\, dx$

Let $I = \int \log x \; dx = \int 1 . \log x \; dx$

Let $u = \log x$, $w = 1$ in the formula for integration of product of two functios,

$$\therefore I = \log x \int 1 \; dx - \int \left[\frac{d}{dx} (\log x) \int 1 \; dx\right] dx$$

$$= \log x \; (x) - \int \left[\left(\frac{1}{x}\right) (x)\right] dx$$

$$= x \log x - \int 1 \; dx = x \log x - x + c$$

Thus, $I = x \log x - x + c$

By Vertically and Crosswise Sūtra,

Let $I = \int \log x \; dx$

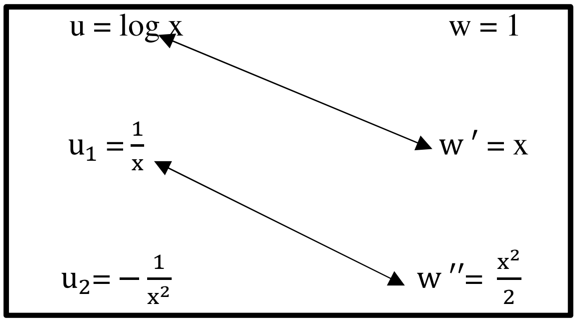


Figure: 5.18

By using,

$$\int \log x \; dx = uw' - u_1 w'' + u_2 w'''$$

$$\therefore \int \log x \; dx = x \log x - \left(\frac{1}{x}\right) \left(\frac{x^2}{2}\right) + c$$

Here, if we take successive differentiation of the term u = log x then it is not possible that at any order the derivative becomes zero.

I.e. by Vertically & Crosswise Sūtra it is not possible to solve the above problem. By using standard formula of integration of multiplication of two functions

$$\int \log x \; dx = x \log x - x + c$$

Thus, Integration problems based on partial fraction can be solved by using Parāvartya Yojayet Sūtra and integration problems based on multiplication of two functions can be solved by Vertically & Crosswise Sūtra, if we get any order of derivative of u as zero.

Power Series representation:

Assuming that g(x) has derivative of every order, a power series representation about x = a is written as

$$\boxed{\begin{aligned} g(x) &= \sum_{n=0}^{\infty} c_n\,(x-a)^n \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots\ldots\ldots\ldots\ldots \end{aligned}}$$

To determine the co-efficient c_n, evaluating above series at x which gives, $f(a) = c_0$

Other co-efficient c_1 can be find out by taking the derivative of the power series. $g'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \ldots\ldots\ldots$

$\therefore g'(a) = c_1$

By taking second, Third and Fourth order derivative coefficients $c_2, c_3, c_4 \ldots\ldots$ can be find out respectively.

$g''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + \ldots\ldots\ldots$

$\therefore g''(a) = 2c_2 \ \therefore c_2 = \dfrac{f''(a)}{2} = \dfrac{f''(a)}{2!}$

Similarly, $c_3 = \frac{g'''(a)}{2(3)} = \frac{g'''(a)}{3!}$;

$c_4 = \frac{g''''(a)}{2(3)(4)} = \frac{g''''(a)}{4!}$;.... In general, $c_n = \frac{g^n(a)}{n!}$

Thus, provided that there exists a power series representation, the Taylor series is as follows.

Taylor series:

$$g(x) = \sum_{n=0}^{\infty} \frac{g^n(a)}{n!}(x-a)^n$$

$$\text{i. e. } g(x) = g(a) + (x-a)\,g'(a) + (x-a)^2\frac{g''(a)}{2!} + (x-a)^3\frac{g'''(a)}{3!} + \ldots\ldots\ldots\ldots$$

Maclaurin series:

In Taylor series substitute x = 0; we get Maclaurin's series

$$g(x) = \sum_{n=0}^{\infty} \frac{g^n(0)}{n!} x^n$$

i.e. $g(x) = g(0) + x\,g'(0) + x^2\frac{g''(0)}{2!} + x^3\frac{g'''(0)}{3!} + \ldots\ldots\ldots\ldots$

Example: 22

Find the series of $f(x) = e^x$ by using Taylor's Series at x = 0.

Taylor Series at x = 0 means Maclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\ldots \text{(a)}$$

$$f(x) = e^x, f'(x) = e^x, f''(x) = e^x, \ldots\ldots\ldots\ldots\ldots\ldots f^n(x) = e^x$$

By putting x = 0 in equation (a).

$$\therefore f(0) = 1, f'(0) = 1, f''(0) = 1, f'''(0) = 1, \ldots\ldots\ldots\ldots f^n(0) = 1$$

$$\therefore f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

By using Taylor Series at x = 0 the following Series can be obtained:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots\ldots\ldots \qquad \therefore \sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots\ldots \qquad \therefore \cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots\ldots\ldots\ldots \therefore \log(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}$$

Example: 23

Find the Taylor series for $f(x) = e^x.\sin x$ about x = 0.

$$f(x) = e^x.\sin x$$

$$= \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots\ldots\ldots\right]$$

By using Vertically & Crosswise Sūtra for finding $f^n(x)$

$$= \left[1 \;+\; x \;+\; \frac{x^2}{2!} \;+ \frac{x^3}{3!} \;+\; \frac{x^4}{4!} \;\ldots\ldots\ldots\right]$$

$$\times$$

$$\left[x \;-\; \frac{x^3}{3!} \;+\; \frac{x^5}{5!} \;-\; \frac{x^7}{7!} \;+\ldots\ldots\ldots\ldots\right]$$

By taking vertically multiplication of first terms, two terms, three terms and continuing in increasing order of both the series,

$$= x + \left[x^2 - \frac{x^3}{3!}\right] + \left[\frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{5!}\right] + \left[\frac{x^5}{4!} + \frac{x^6}{(3!)^2} + \frac{x^7}{2!\,5!} - \frac{x^7}{7!}\right] + \ldots\ldots\ldots\ldots$$

$$\therefore f(x) = x + x^2\left[\frac{1}{2!} - \frac{1}{3!}\right]x^3 - \frac{1}{3!}x^4 + \left[\frac{1}{5!} + \frac{1}{4!}\right]x^5 + \frac{1}{(3!)^2}x^6 + \left[\frac{1}{2!\,5!} - \frac{1}{7!}\right]x^7 + \ldots\ldots$$

The above equation is the Taylor series of $e^x.\sin x$ at x = 0.

Example: 24

Find the Taylor Series of $f(x) = x^4e^{3x}$

$$f(x) = x^4e^{3x} = x^4\sum_{n=0}^{\infty}\frac{(3x)^n}{n!} = x^4\sum_{n=0}^{\infty}\frac{(3)^n(x)^n}{n!}$$

$\therefore f(x) = \sum_{n=0}^{\infty}\frac{(3)^n(x)^{n+4}}{n!}$ is the Taylor Series for $f(x) = x^4e^{3x}$ at x = 0.

Application of Taylor Series for finding Integration:

Example: 25

Determine a Taylor series about x = 0 for the Integral $\int \frac{\cos x}{x}$

First find the Taylor series for the Integrand.

$$\therefore \int \frac{\cos x}{x} = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)(2n)!}$$

5.9 Solving DE using method of series:

To find the solution of the type linear & non-linear differential equations ordinary as well as partial by using particular method of series.

Also differential integral, Integro-differential equations, double integration can be easily solved by using Power series, Taylor series and Maclaurin series.

Another way to find out Series Expansion:

Example: 26

Expand $(1 + x)^5$ as a power series.

Let $(1 + x)^5 = A + B x + Cx^2 + Dx^3 + Ex^4 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ [1]

Taking log and then derivative on both the sides w.r.t. x,

$5\log(1 + x) = \log(A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots\ldots)$

Taking derivative on both the sides,

5 20 30 20 5

$$\therefore \frac{5}{1+x} = \frac{B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4}{A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5} \ldots\ldots\ldots..[2]$$

1 5 10 10 5 1

Substitute the value of x as zero in equation [1], we get A = 1.

By using **Vertically & Crosswise Sūtra** in [2] and then comparing the co-eff. of x, x^2, x^3 ….. Corresponding respected value of B, C and D can be obtained.

By comparing constant term, 5A = B but A = 1 ∴ B = 5;

By comparing coefficient of x, 5B = B + 2C ∴ 4B = 2C ∴ 2C = 20 ∴ C = 10;

By comparing coefficient of x^2, 5C = 3D + 2C ∴ 3D = 3C ∴ 3D = 30 ∴ D =10;

By comparing coefficient of x^3, 5D = 4E + 3D ∴ 4E = 2D ∴ 4E = 20 ∴ E = 5:

Continuing this way put the already founded values in equation [1]

The series $(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$ is obtained.

5.9.1 Linear Differential Equations by using alternate elimination and retention:

Example: 27

Solve the following Linear Equation:

$y' + 2y + 6 = 4x + 16$, where $y(0) = 4$………. (1)

By Maclaurin's theorem a series expansion can be used for y, as an equation

$y = a + bx + cx^2 + dx^3 + ex^4 + \ldots\ldots\ldots..$ (2)

Differentiating equation (2) gives equation (3), successive coefficients of x being arranged in successive columns.

$y' = b + 2cx + 3dx^2 + 4ex^3 + \ldots\ldots\ldots\ldots\ldots\ldots..$ (3)

Coefficients	column
	1 2 3 4
	4 2 -3 0
2	$y = a + bx + cx^2 + dx^3 + ex^4 + \ldots\ldots\ldots$ Row 1
1	$y' = b + 2cx + 3dx^2 + 4ex^3 + \ldots\ldots\ldots\ldots$ Row 2

$$\begin{array}{cccc} 2 & -6 & 0 & 0 \\ \hline 16 & 4 & & \end{array} \quad \text{Row 3}$$

The left hand side of equation (1) translate into Row 2 + 2 × Row 1+ 6.

The right hand side terms of equation (1) have been placed in third row.

Thus, for each successive column we have Row 2 + 2 × Row 1 + 6 = Row 3 in accordance with equation (1).

An each successive column yields the next of unknowns a, b, c, d…here we use the Lopanasthāpanābhyām Sub-Sūtra.

It is given that $y(0) = 4$, therefore from row 1, $a = 4$.

To find b, comparing constant term, column 1 gives, $b + 2a + 6 = 16$

Substituting $a = 4$, $b + 2(4) + 6 = 16$

$$\therefore B + 8 + 6 = 16 \therefore b + 14 = 16 \therefore b = 16 - 14 = 2 \therefore b = 2$$

To find c, comparing co-efficient of x, column 2 gives, $2c + 2b + 6 = 4$,

Substituting $b = 2$,

$$\therefore 2c + 2(2) + 6 = 4 \therefore 2c + 4 + 6 = 4 \therefore 2c = 4 - 10 \therefore 2c = -6 \therefore c = -3$$

To find d comparing co-efficient of x^2, column 3 gives, $3d + 2c + 6 = 0$,

Substituting $c = -3$,

$\therefore 3d + 2(-3) + 6 = 0 \therefore 3d - 6 + 6 = 0 \therefore 3d = 0 \therefore d = 0$

The same procedure shows that all subsequent terms are zero, and the solution,

$y = a + b x + c x^2 + dx^3 + ex^4 + \ldots$ is $\mathbf{y = 4 + 2x - 3x^2}$ exactly.

The complete working and solution appears finally as follows:

$$\begin{array}{l} \quad 4 \quad 2 \quad -3 \quad 0 \quad 0 \\ y = a + bx + cx^2 + dx^3 + ex^4 + \ldots\ldots\ldots\ldots \quad (2) \\ y' = b + 2cx + 3dx^2 + 4ex^3 + \ldots\ldots\ldots\ldots \quad (3) \\ \quad 2 \quad -6 \quad 0 \quad 0 \end{array}$$

I.e. The solution is **$y = 4 + 2x - 3x^2$**

Example: 28

Solve the following Linear Equation

$y'' + 2y' + 6y = 2 - 30x^2$ given that $y(0) = 1$, $y'(0) = 4$…………….. (1)

1 4 -6 0 0 0

$y = a + bx + cx^2 + dx^3 + ex^4 + fx^5$……………………………………… (2)

4 -12 0 0 0

$y' = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4$……………………………………… (3)

-12 0 0 0

$y'' = 2c + 6dx + 12ex^2 + 20fx^3$…………………………………………... (4)

$2 - 30x^2$

Since $y(0) = 1$, $\boxed{a = 1}$ and $y'(0) = 4$, $\boxed{b = 4}$

To find c,

Comparing constant term gives, $2c + 2b + 6a = 2$, Substituting $a = 1$ and $b = 4$

$\therefore 2c + 2(4) + 6(1) = 2 \therefore 2c + 14 = 2$ i.e. $\boxed{c = -6}$

To find d,

Comparing the co-eff. of x gives, $6d + 4c + 6b = 0$, Substituting $b = 4$ and $c = -6$

$\therefore 6d + 4(-6) + 6(4) = 0 \therefore 6d - 24 + 24 = 0 \therefore 6d = 0$ $\boxed{\therefore d = 0}$

To find e,

Comparing the co-eff. of x^2 gives, $12e + 6d + 6c = -30$,

Substituting $c = -5$, $d = 0$

$\therefore 12e + 6(0) + 6(-5) = -30 \therefore 12e + 0 - 30 = -30 \therefore 12e = 0$ $\boxed{e = 0}$

To find f,

Comparing the co-eff. of x^3 gives, $20f + 8e + 6d = 0$,

Substituting $e = 0$, $d = 0$

$\therefore 20f + 8(0) + 6(0) = 0 \therefore 20f = 0 \therefore$ $\boxed{f = 0}$

Substituting the value of the above values a = 1, b = 4, c = - 6, d = 0, e = 0, f = 0…in equation (2).

Thus, the solution is $y = 1 + 4x - 6x^2$ exactly.

Example: 29

$4y' + 7y + 2 = 2 + 23\exp(4x) - 11\exp(x)$

$y = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \ldots\ldots\ldots$ [1]

$y' = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4 + \ldots\ldots\ldots\ldots$ [2]

$$\exp(x) = e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots$$

$$\therefore \exp(4x) = e^{4x} = 1 + 4x + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \frac{(4x)^4}{4!} + \ldots\ldots\ldots$$

$$\therefore e^{4x} = 1 + 4x + \frac{16x^2}{2!} + \frac{64x^3}{3!} + \frac{256x^4}{4!} + \ldots\ldots\ldots$$

R.H.S.

$= 2 + 23\exp(4x) - 11\exp(x)$

$$= 2 + 23\left(1 + 4x + \frac{16x^2}{2!} + \frac{64x^3}{3!} + \frac{256x^4}{4!} + ..\right) - 11\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ..\right)$$

$$= 2 + (23 - 11) + (184 - 11)x + (184 - 5.5)x^2 + (245.33 - 3.67)x^3 + \ldots..$$

$$= 14 + 173x + 178.5x^2 + 241.66x^3 + \ldots\ldots\ldots\ldots$$

L.H.S.

$= 4y' + 7y + 2$

$= 4[b + 2cx + 3dx^2 + 4ex^3 + 5fx^4 + \ldots.] + 7[a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \ldots\ldots] + 2$

$= (4b + 7a + 2) + (8c + 7b)x + (12d + 7c)x^2 + (16e + 7d)x^3 + (20f + 7e)x^4 + \ldots\ldots\ldots$

Comparing the constant term and co-eff. of x, x^2, x^3………

$4b + 7a + 2 = 14, a = 0 \therefore 4b = 12$

$$\therefore b = 3$$

$8c + 7b = 173, b = 3 \therefore 8c + 21 = 173 \therefore 8c = 173 - 21 = 152$

$$\therefore c = \frac{152}{8} = 19$$

$12d + 7c = 178.5 \therefore 12d + 7(19) = 178.5 \therefore 12d = 178.5 - 133 = 45.5$

$$\therefore d = \frac{45.5}{12} = 3.79$$

$16e + 7d = 241.66 \;\therefore\; 16e + 7(3.79) = 241.66 \;\therefore 16e + 26.54 = 241.66$

$\therefore 16e = 241.66 - 26.54 = 215.12$

$$\therefore e = \frac{215.12}{16} = 13.45$$

By substituting value of a, b, c, d, e …in the equation [1]

∴The solution is $y = 3x + 19x^2 + 3.79x^3 + 13.45x^4 + \ldots\ldots\ldots$

5.9.2 Using alternate elimination & retention Sūtra for DE of non linear type:

By using alternate elimination and retention Sūtra along with power series solution of problems of differential equation in non-linear form effortlessly.

Example: 30

Solve the following non-linear Equation:

$2y + (y')^2 = 13 + 18x + 6x^2$, where y (1) = 6

The boundary conditions not being given at the origin, we can expand y as a series in (x - 1). We have,

$2y + (y')^2 = 37 + 30(x - 1) + 6(x - 1)^2$, where y (1) = 6

Let (x – 1) = m

Therefore the new expression in terms of m will be

$y = A + Bm + Cm^2 + Dm^3 + Em^4 + Fm^5 + \ldots\ldots\ldots$

$\therefore y' = B + 2Cm + 3Dm^2 + 4Em^3 + 5Fm^4 + \ldots\ldots$

RHS = $37 + 30m + 6m^2$

Since y (1) is given, A = 6

To find B, comparing constant term, $2A + B^2 = 37$, substituting A = 6

$\therefore 2(6) + B^2 = 37 \therefore 12 + B^2 = 37 \therefore B^2 = 37 - 12 = 25 \therefore B = \pm 5$

This leads to two solution, one with B = +5 and one with B = -5

Enclosing the first solution by horizontal line, the second solution is written down, outside them, thus

$$\begin{array}{cccccl}
6 & -5 & -2 & 0.2 & -0.11 & \ldots\ldots\ldots\ldots\ldots\ldots \\
\hline
6 & 5 & 1 & 0 & 0 &
\end{array}$$

$$y = A + Bm + Cm^2 + Dm^3 + Em^4 + Fm^5 + \ldots\ldots\ldots$$

$$y' = B + 2Cm + 3Dm^2 + 4Em^3 + 5Fm^4 + \ldots\ldots\ldots$$

$$\begin{array}{cccccl}
+5 & 2 & 0 & 0 & 0 & \ldots\ldots\ldots\ldots\ldots\ldots \\
\hline
-5 & -4 & 0.6 & -0.44 & &
\end{array}$$

The two solutions are:

$y = 6 + 5(x - 1) + (x - 1)^2$ (exact solution)

$y = 6 - 5(x - 1) - 2(x - 1)^2 - 2(x - 1)^3 - 0.11(x - 1)^4 \ldots\ldots\ldots$

Explanation:

For A = 6 and B = 5

Comparing coefficient of (x - 1)

$2B + 4BC = 30$, substituting B = 5,

$\therefore 2(5) + 4(5)C = 30$

$\therefore 10 + 20C = 30 \therefore 20C = 20$

$\therefore C = 1$

Comparing coefficient of $(x - 1)^2$

$2C + 4C^2 + 6BD = 6$, substituting B = 5 and C = 1,

$\therefore 2(1) + 4(1)^2 + 6(5)D = 6$

$\therefore 2 + 4 + 30D = 6 \therefore 30D = 0$

$\therefore D = 0$

Comparing coefficient of $(x - 1)^3$

$2D + 12CD + 8BE = 0$, substituting B = 5, C = 1 and D = 0

$\therefore 2(0) + 12(1)(0) + 8(5)E = 0 \therefore 0 + 40E = 0$

$\therefore E = 0$

The solution is $y = 6 + 5(x - 1) + (x - 1)^2$ (exact solution)

In the same way,

For A = 6 and B = -5,

Comparing coefficient of $(x - 1)$

$2B + 4BC = 30$, substituting $B = -5$,

$\therefore 2(-5) + 4(-5)C = 30$

$\therefore -10-20C = 30 \therefore -20C = 40$

$\therefore C = -2$

Comparing coefficient of $(x-1)^2$

$2C + 4C^2 + 6BD = 6$, substituting $B = - 5$ and $C = - 2$,

$\therefore 2(-2) + 4(-2)^2 + 6(-5)D = 6$

$\therefore -4 +16 - 30D = 6$

$\therefore -30D = 6-12$

$\therefore D = 0.2$

Thus, $y = 6 - 5(x - 1) - 2(x - 1)^2 - 2(x - 1)^3 - 0.11(x - 1)^4$…….

Example: 31

Solve the following non-linear equation:

$y + \exp(y') = 1 + 3x + 2x^2$ given $y'(0) = 0$………………… [1]

Letting $y = a + b x + cx^2 + \ldots\ldots$

$\therefore y' = b + 2cx + 3dx^2 + 4ex^3 + \ldots$

Equation [1] can be rewritten as $\exp(y') = 1 + 3x + 2x^2 - y$

$\exp(b + 2cx + 3dx^2 + 4ex^3 + \ldots\ldots.)$

$= 1 + 3x + 2x^2 - y$

$= 1 + 3x + 2x^2 - (a + b x + cx^2 + \ldots\ldots)$

$= (1 - a) + (3 - b) x + (2 - c) x^2 - dx^3 - ex^4 - \ldots\ldots\ldots..$ [2]

Differentiating Equation [2] w.r.t. x and dividing by equation [2],

$3 \quad 3 \quad 26\frac{1}{2} \quad -88c \quad 29\frac{1}{9}$

$2c + 6dx + 12ex^2 + 20fx^3 + 30gx^4 + \ldots\ldots\ldots$

$3 \quad 1 \quad 4 \quad -8\frac{5}{6} \quad 22\frac{1}{12} \quad -58\frac{9}{40}$

$$= \frac{(3-b)+(4-2c)x-3dx^2-4ex^3-5fx^4-6gx^5 \ldots\ldots\ldots\ldots}{(1-a)+(3-b)x-(2-c)x^2-dx^3-ex^4-fx^5-gx^6 \ldots\ldots} \ldots\ldots [3]$$

$1 \quad 3 \quad \frac{1}{2} \quad \frac{4}{3} \quad -2\frac{5}{24} \quad 4\frac{5}{12} \quad -9\frac{169}{240}$

Explanation:

Firstly, b = 0 from boundary conditions, and then:

From equation [2] by comparing coefficient of unity a = 1.

From equation [3]

Comparing coefficient of unity $2c = 3 \therefore c = \frac{3}{2}$

Comparing coefficient of x, $6d = -8 \therefore d = -\frac{4}{3}$

Comparing coefficient of x^2, $12e = 26\frac{1}{2} \therefore e = \frac{53}{24} \therefore e = 2\frac{5}{24}$

Comparing coefficient of x^3, $20f = -88\frac{1}{3} - 8\frac{5}{6} - 26\frac{1}{2}x^3 = -88\frac{1}{3}$

$$\therefore f = -\frac{265}{60} = -4\frac{25}{60} = -4\frac{5}{12}$$

Comparing coefficient of x^4,

$$30g = 22\frac{1}{2} + 6\frac{5}{8} + 10\frac{2}{3} = 13\frac{1}{4} + 265 = 1\frac{1}{8}$$

$$\therefore g = \frac{2329}{240} = 9\frac{169}{240}$$

Substitute the values of seven terms in the solution

$y = a + b\,x + cx^2 + d\,x^3 + ex^4 + fx^5 + gx^6 + \ldots\ldots$

We get, $y = 0 + 0\,x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + 2\frac{5}{24}x^4 - 4\frac{5}{12}x^5 + 9\frac{169}{240}x^6 + \ldots\ldots\ldots.$

Therefore, $y = \frac{3}{2}x^{2} - \frac{4}{3}x^{3} + 2\frac{5}{24}x^{4} - 4\frac{5}{12}x^{5} + 9\frac{169}{240}x^{6} + \ldots\ldots\ldots.$

5.10 Integro-Differential Equation:

5.10.1 Integral equation:

Example: 32

$$y^2 + \left(\frac{dy}{dx}\right)^2 + y\int_0^x y\,dx = 8 + 34x + 54x^2 + 18x^3 + 34x^4 + 91x^5$$

$y = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + \ldots\ldots..$

$$\therefore y^2 = a^2 + 2abx + 2acx^2 + b^2x^2 + 2adx^3 + 2bcx^3 + 2aex^4 + 2bdx^4$$
$$+ c^2x^4 + 2afx^5 + 2bex^5 + 2cdx^5 + \ldots.$$

$$\therefore \left(\frac{dy}{dx}\right) = y' = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4 + 6gx^5 + \ldots\ldots.$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = b^2 + 4bcx + 6bdx^2 + 4c^2x^2 + 8bex^3 + 12cdx^3 + 10bfx^4 + 16cex^4$$
$$+ 9d^2x^4 + 12bgx^5 + 20cfx^5 + 12dex^5 + \ldots\ldots\ldots..$$

$\therefore$ Now, $y = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + \ldots\ldots..$

$$\int_0^x y\,dx = ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 + \frac{d}{4}x^4 + \frac{e}{5}x^5 + \ldots\ldots\ldots..$$

By using Vertically & Crosswise Sūtra,

$$y\int_0^x y\,dx = a^2\,x + \frac{ab}{2}x^2 + abx^2 + acx^3 + \frac{ac}{3}x^3 + \frac{b^2}{2}x^3 + \frac{ad}{4}x^4 + adx^4$$

$$+\frac{bc}{3}x^4+\frac{bc}{2}x^4+\frac{ae}{5}x^5+aex^5+\frac{bd}{4}x^5+\frac{bd}{2}x^5+\frac{c^2}{3}x^5+\ldots\ldots\ldots$$

$$\text{L. H. S.} = \quad y^2+\left(\frac{dy}{dx}\right)^2+y\int_0^x ydx$$

$$= a^2+b^2+[2c+4bc+a^2]x+\left[2ac+b^2+6bd+4c^2+\frac{ab}{2}\right]x^2$$

$$+\left[2ad+2bc+8be+12cd+\frac{4}{3}ac+\frac{b^2}{2}\right]x^3$$

$$+\left[2ae+2bd+c^2+10bf+16ce+9d^2+\frac{5}{4}ad+\frac{5}{6}bc\right]x^4$$

$$+\left[2af+2be+2cd+12bg+20cf+12de+\frac{6}{5}ae+\frac{3}{4}bd+\frac{c^2}{3}\right]x^5+\cdots$$

$\text{R. H. S.} = 13+28x+22x^2+173x^3+3x^4+45x^5$

Comparing coefficient on both sides,

$y(0)=2 \therefore a=2$ and $a^2+b^2=8 \therefore b=\pm 2$

For $b=2$,

$2c+4bc+a^2=34$

$\therefore 2c+8c+4=34$

$\therefore 10c=30$

$\therefore c=3$

$2ac+b^2+6bd+4c^2+\frac{ab}{2}=54$

$\therefore 12+4+12d+36+2=54$

$\therefore d=0$

$2ad+2bc+8be+12cd+\frac{4}{3}ac+\frac{b^2}{2}=18$

$\therefore 8+16e+8+2=18$

$\therefore e=0$

$2ae + 2bd + c^2 + 10bf + 16ce + 9d^2 + \frac{5}{4}ad + \frac{5}{6}bc = 34$

$\therefore 9 + 20f + 5 = 34 \therefore 20f = 34 - 14$

$\therefore 20f = 20$

$\therefore f = 1$

$2af + 2be + 2cd + 12bg + 20cf + 12de + \frac{6}{5}ae + \frac{3}{4}bd + \frac{c^2}{3} = 91$

$4 + 24g + 60 + 3 = 91 \therefore 24g = 91 - 67 = 24$

$\therefore 24g = 24$

$\therefore g = 1$

$\therefore \quad \mathbf{y = 2 + 2x + 3x^3 + x^5 + x^6\ (for\ \ b = 2)}$

For $b = -2,$

$2c + 4bc + a^2 = 34$

$\therefore 2c - 8c + 4 = 34$

$\therefore -6c = 30$

$\therefore c = -5$

$2ac + b^2 + 6bd + 4c^2 + \frac{ab}{2} = 54$

$\therefore -20 + 4 - 12d + 100 - 2 = 54$

$\therefore -12d = 54 - 82$

$\therefore d = 2.33$

$2ad + 2bc + 8be + 12cd + \frac{4}{3}ac + \frac{b^2}{2} = 18$

$\therefore 9.32 + 20 - 16e - 139.8 + 13.33 + 2 = 18$

$\therefore -16e = 18 + 95.15 = 113.5$

$\therefore e = 7.07$

$2ae + 2bd + c^2 + 10bf + 16ce + 9d^2 + \frac{5}{4}ad + \frac{5}{6}bc = 34$

$\therefore 28.28 - 9.32 + 25 - 20f - 70.7 + 48.86 + 5.83 + 8.33 = 34$

$\therefore -20f = 34 - 35.78$

$\therefore -20f = -1.78$

$\therefore f = 0.089$

$$2af + 2be + 2cd + 12bg + 20cf + 12de + \frac{6}{5}ae + \frac{3}{4}bd + \frac{c^2}{3} = 91$$

$0.356 - 28.28 - 23.3 - 24g - 8.9 + 197.68 + 16.97 - 8.74 + 8.33 = 91$

$\therefore -24g = 91 - 154.116$

$\therefore -24g = -63.116$

$\therefore g = 2.63$

Therefore, another solution is

$$\mathbf{y = 2 - 2x - 5x^2 + 2.33x^3 + 7.07x^4 + 0.089x^5 + 2.63x^6 (for \quad b = -2)}$$

5.10.2 Double Integration:

Example: 33

$$\int_0^x \int_0^x y^2 dxdx + \int_0^x ydx = 2x + \frac{7}{2}x^2 + 2x^3 + \frac{3}{4}x^4$$

Let $y = a + bx + cx^2 + dx^3 + ex^4 + \ldots\ldots\ldots\ldots\ldots [1]$

$$\therefore \int_0^x ydx = ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 + \frac{d}{4}x^4 + \frac{e}{5}x^5 \ldots\ldots [2]$$

$$y^2 = a^2 + 2abx + 2acx^2 + b^2x^2 + 2bcx^3 + 2adx^3 + 2aex^4 + 2bdx^4 + c^2x^4 + ..$$

$$\therefore \int_0^x y^2 dx = a^2x + ab\,x^2 + \frac{2}{3}acx^3 + \frac{1}{3}b^2x^3 + \frac{1}{2}bcx^4 + \frac{1}{2}adx^4$$
$$+ \frac{2}{5}aex^5 + \frac{2}{5}bdx^5 + \frac{2}{5}c^2x^5 \ldots\ldots\ldots..$$

$$\therefore \int_0^x \int_0^x y^2 dxdx = \frac{a^2}{2}x^2 + \frac{1}{3}abx^3 + \frac{1}{6}acx^4 + \frac{1}{12}b^2x^4 + \frac{1}{10}bcx^5 + \frac{1}{10}adx^5 + \cdots [3]$$

By adding equation [3] & [2] we get left side of the given integral.

Comparing it with the coefficient of the right side.

Comparing the coefficient of x: $a = 2,$

Comparing coefficient of x^2:

$$\left[\frac{b}{2} + \frac{a^2}{2}\right] = \frac{7}{2}$$

$\therefore b + a^2 = 7 \therefore b + 4 = 7$

$\therefore b = 3$

Comparing the coefficient of x^3:

$$\left[\frac{c}{3} + \frac{ab}{3}\right] = 2$$

$\therefore c + ab = 6$

$\therefore c + 2(3) = 6$

$\therefore c = 0$

Comparing the coefficient of x^4:

$$\left[\frac{d}{4} + \frac{ac}{6} + \frac{b^2}{12}\right] = \frac{3}{4}$$

$$\therefore \left[\frac{d}{4} + \frac{2(0)}{6} + \frac{9}{12}\right] = \frac{3}{4}$$

$$\therefore \frac{d}{4} = \frac{3}{4} - \frac{3}{4} = 0$$

$\therefore d = 0$

Substituting the values of a, b, c, d……….in equation [1]

$y = 2 + 3x + 0\,x^2 + 0\,x^3 + \ldots\ldots..$

$\therefore$ The solution is $y = 2 + 3x$

Partial Derivative:

If any function contains more than two variables then partial derivative can be found by taking derivative w.r.t. one variable keeping others held constant. Partial Derivatives are used in Vector Calculus and Differential geometry.

First & Second Order Partial Derivatives:

Let z be expressed in terms of x, y, z, etc

Keeping variable y constant;

$$\frac{\partial z}{\partial x} = f_x$$

Keeping variable x constant;

$$\frac{\partial z}{\partial y} = f_y;$$

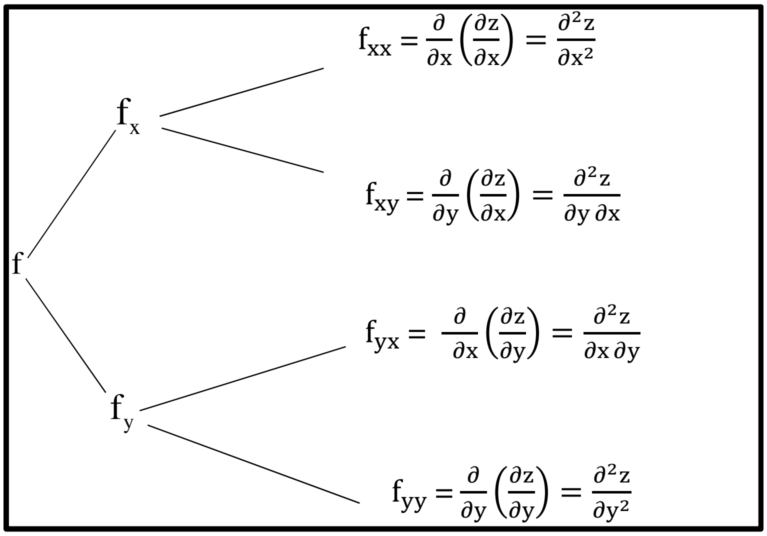
By using First order partial order derivative f_x and f_y above,

Second order partial derivative w.r.t x and y can be obtained.

Partial derivatives of f_x w.r.t. x i.e. f_{xx} & f_{xy} and

Partial derivative of w.r.t. y i.e. f_{yx} & f_{yy} are explained with the following figure:

Second order Partial Derivative:



f
f_x
f_y
$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$
$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x}$
$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y}$
$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$

Figure: 5.19

5.11 Partial Differential Equations:

Functions polynomials with single Variable can be expressed as a Power Series same way we can express functions of bi-polynomials with two variables whose applications are explained:

Let U = U(x, y)

$$U = a_0 + b_0 x + c_0 x^2 + d_0 x^3 + e_0 x^4 + \ldots\ldots\ldots\ldots\ldots..$$
$$a_1 y + b_1 xy + c_1 x^2 y + d_1 x^3 y + e_1 x^4 y + \cdots$$

$$a_2y^2 + b_2xy^2 + c_2x^2y^2 + d_2x^3y^2 + e_2x^4y^2 + \cdots$$

$$a_3y^3 + b_3xy^3 + c_3x^2y^3 + d_3x^3y^3 + \ldots$$

$$a_4y^4 + b_4xy^4 + c_4x^2y^4 + d_4x^3y^4 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad [1]$$

$$\therefore U\,(0,0) = a_0$$

$$U_x = b_0 + 2c_0x + 3d_0x^2 + 4e_0x^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$b_1y + 2c_1xy + 3d_1x^2y + 4e_1x^3y + \ldots\ldots\ldots\ldots\ldots$$

$$b_2y^2 + 2c_2xy^2 + 3d_2x^2y^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$b_3y^3 + 2c_3xy^3 + 3d_3x^2y^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$b_4y^4 + 2c_4xy^4 + 3d_4x^2y^4 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [2]$$

$$\therefore U_x(0,0) = b_0$$

$$U_{xx} = 2c_0 + 6d_0x + 12e_0x^2 + \cdots 2c_1y + 6d_1xy + 12e_1x^2y + \ldots\ldots$$

$$+ 2c_2y^2 + 6d_2xy^2 + \ldots\ldots\ldots\ 2c_3y^3 + 6d_3xy^3 + \ldots\ldots\ldots\ldots\ldots$$

$$+ 2c_4y^4 + 6d_4xy^4 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [3]$$

$$\therefore U_{xx}(0,0) = 2c_0$$

$$U_{xxx} = U_{x^3} = 6d_0 + 24e_0x^2 + 20f_0x^3 + \ldots\ldots\ldots$$

$$+6d_1y + \ldots\ldots +6d_2y^2 + \ldots +6d_3y^3 + \ldots\ldots\ldots\ldots\ldots\ldots [4]$$

$$\therefore U_{x^3}(0,0) = 6d_0$$

$$U_y = a_1 + b_1x + c_1x^2 + d_1x^3 + e_1x^4 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$2a_2y + 2b_2xy + 2c_2x^2y + 2d_2x^3y + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$3a_3y^2 + 3b_3xy^2 + 3c_3x^2y^2 + 3d_3x^3y^2 + \ldots\ldots\ldots\ldots$$

$$4a_4y^3 + 4b_4xy^3 + 4c_4x^2y^3 + 4d_4x^3y^3 + \ldots\ldots\ldots\ldots\ldots\ldots [5]$$

$$\therefore U_y(0,0) = a_1$$

$$U_{yy} = 2a_2 + 2b_2x + 2c_2x^2 + 2d_2x^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$6a_3y + 6b_3xy + 6c_3x^2y + 6d_3x^3y + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$12a_4y^2 + 12b_4xy^2 + 12c_4x^2y^2 + 12d_4x^3y^2 + \ldots\ldots\ldots\ldots [6]$$

$$\therefore U_{yy}(0,0) = 2a_2$$

$U_{yyy} = U_{y^3} = 6a_3 + 6b_3x + 6c_3x^2 + 6d_3x^3 + \ldots\ldots$

$24a_4y + 24b_4xy + 24c_4x^2y + 24d_4x^3y + \ldots\ldots\ldots [7]$

$\therefore U_{yyy}(0,0) = 6a_3$

$U_{xy} = b_1 + 2c_1x + 3d_1x^2 + 4e_1x^3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$

$2b_2y + 4c_2xy + 6d_2x^2y + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$3b_3y^2 + 6c_3xy^2 + 9d_3x^2y^2 + \ldots\ldots\ldots\ldots\ldots\ldots..$

$4b_4y^3 + 8c_4xy^3 + 12d_4x^2y^3 + \ldots\ldots\ldots\ldots\ldots\ldots [8]$

$\therefore U_{xy}(0,0) = b_1$

$U_{yx} = b_1 + 2c_1x + 3d_1x^2 + 4e_1x^3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots$

$2b_2y + 4c_2xy + 6d_2x^2y + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

$3b_3y^2 + 6c_3xy^2 + 9d_3x^2y^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

$4b_4y^3 + 8c_4xy^3 + 12d_4x^2y^3 + \ldots\ldots\ldots\ldots\ldots\ldots [9]$

$\therefore U_{yx}(0,0) = b_1$

Similarlly,

$U_{xy^2}(0,0), U_{xy^3}(0,0), \ldots\ldots\ldots\ldots..$ and $U_{yx^2}(0,0), U_{yx^3}(0,0)$ can be obtained.

5.11.1 PDE of linear type:

Example: 34

$U_{x^3} + U_{y^2} + U_{xy} = 12$

$U(0,0) = 4, U_x(0,0) = 3, U_{x^2}(0,0) = 4, U_{x^3}(0,0) = 6, U_{x^4}(0,0) = \ldots\ldots = 0$

and $U_y(0,0) = 2, U_{y^2}(0,0) = 2, U_{y^3}(0,0) = \ldots\ldots\ldots 0,$

$U_{xy}(0,0) = 4, U_{x^2y}(0,0) = \ldots\ldots\ldots\ldots = 0$

For given boundary conditions by using equations [1] to [9],

$U(0,0) = 4 = a \ \therefore a_0 = 4; U_x(0,0) = 3 = b_0 \ \therefore b_0 = 3;$

$U_{x^2}(0,0) = 4 = 2c_0 \ \therefore 2c_0 = 4 \ \therefore c_0 = 2$

$$U_{x^3}(0,0) = 6 = 6d_0 \;\therefore 6d_0 = 6 \;\therefore d_0 = 1$$

$$U_{x^4}(0,0) = \ldots\ldots\ldots = 0 \;\therefore e_0 = f_0 = \ldots\ldots\ldots = 0$$

$$U_y(0,0) = 2 = a_1 \therefore a_1 = 2; U_{y^2}(0,0) = 2 = 2a_2 \;\therefore a_2 = 1$$

$$U_{y^3}(0,0) = \ldots\ldots\ldots 0 \;\therefore a_3 = a_4 = \ldots\ldots\ldots 0$$

$$U_{xy}(0,0) = 4 \;\therefore b_1 = 4; U_{x^2y}(0,0) = \ldots . = 0$$

$$\therefore c_1 = c_2 = \cdots .0$$

$U_{x^3} + U_{y^2} + U_{xy} = 12$

Comparing the constant term on both the sides,

$\therefore [6d_0 + 2a_2 + b_1] = 12$

$\therefore [6(1) + 2(1) + 4] = 12 \;\therefore 12 = 12$

Comparing the coefficient of $x, x^2 \ldots\ldots\ldots, y, y^2 \ldots\ldots. xy, . x^2y \ldots\ldots..$

$\therefore 24e_0 + 2b_2 + 2c_1] = 0$

$\therefore [24(0) + 2(0) + 2(0)] = 0 \therefore 0 = 0$ ……..

Substituting the obtained values in equation [1], we get

$$\begin{aligned} U = \; & a_0 + b_0x + c_0x^2 + d_0x^3 + e_0x^4 + \ldots\ldots\ldots\ldots\ldots\ldots \\ & a_1y + b_1xy + c_1x^2y + d_1x^3y + e_1x^4y + \ldots\ldots\ldots\ldots \\ & a_2y^2 + b_2xy^2 + c_2x^2y^2 + d_2x^3y^2 + \ldots\ldots\ldots\ldots\ldots.. \\ & a_3y^3 + b_3xy^3 + c_3x^2y^3 + d_3x^3y^3 + \ldots\ldots\ldots\ldots\ldots.. \\ & a_4y^4 + b_4xy^4 + c_4x^2y^4 + d_4x^3y^4 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. [1] \end{aligned}$$

The solution is $U = x^3 + 2x^2 + 3x + 4 + y^2 + 2y + 4xy$

Example: 35

$U_{x^2} + 6xU_y + U_{y^2} = 4$

Given $U(0,0) = 4, U_x(0,0) = 2, U_{x^2}(0,0) = 2, U_{x^3}(0,0) \ldots\ldots\ldots.. = 0$

and $U_y(0,0) = 0, U_{yx}(0,0) = 3, U_{yx^2}(0,0) = \ldots\ldots\ldots\ldots. = 0$

First find $U_x, U_{x^2}, U_{x^3}, U_y, U_{y^2}, U_{yx}, U_{yx}, U_{yx^2}$ at $x = 0$ & $y = 0$ & by given

$U(0,0) = 4 \therefore a_0 = 4; U_x(0,0) = 2 \therefore b_0 = 2$

$U_{x^2}(0,0) = 2 \therefore 2c_0 = 2 \therefore c_0 = 1; U_{x^3}(0,0) \ldots\ldots\ldots. = 0 \therefore d_0 = e_0 = \ldots.0$

$U_y(0,0) = 0 \therefore a_1 = 0, U_{yx}(0,0) = 3 \therefore b_1 = 3$

$U_{yx^2}(0,0) = \ldots\ldots\ldots\ldots = 0 \therefore c_1 = d_1 = 0$

Comparing the constant term on both the sides,

$\therefore [2c_0 + 2a_2] = 4$

$\therefore [2(1) + 2a_2] = 4 \;\; \therefore 2a_2 = 2 \;\; \therefore a_2 = 1$

Comparing the coefficient of $x, x^2 \ldots\ldots\ldots., y, y^2 \ldots\ldots. xy, . x^2y \ldots\ldots\ldots$

$[6d_0 + 2b_2 + 6a_1] = 0 \;\; \therefore [6(0) + 2b_2 + 6(0)] = 0$

$\therefore 2b_2 = 0 \quad \therefore b_2 = 0\ldots\ldots\ldots$ coeff of x

$[12e_0 + 6b_1 + 2c_2] = 0 \;\; \therefore [12(0) + 6(3) + 2c_2] = 0$

$\therefore 2c_2 = 0 \quad \therefore c_2 = 0\ldots\ldots$ coeff of x^2 & so on.

Continuing this way & by substituting all the obtained value in

$$\begin{aligned} U = \; & a_0 + b_0x + c_0x^2 + d_0x^3 + e_0x^4 + \cdots \ldots\ldots\ldots\ldots\ldots. \\ & a_1y + b_1xy + c_1x^2y + d_1x^3y + e_1x^4y + \cdots \ldots\ldots.. \\ & a_2y^2 + b_2xy^2 + c_2x^2y^2 + d_2x^3y^2 + \cdots \ldots\ldots\ldots\ldots. \\ & a_3y^3 + b_3xy^3 + c_3x^2y^3 + d_3x^3y^3 + \cdots \ldots\ldots\ldots\ldots. \\ & a_4y^4 + b_4xy^4 + c_4x^2y^4 + d_4x^3y^4 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots. [1] \end{aligned}$$

The solution is U = x + 3

5.11.2 Solution of nonlinear Partial differential equation:

Example: 36

$U^2 + U_x + U_y = 30 - 10xy + 5x^2y^2 + 2x^2 - 7xy^2$

Given that $U(0,0) = 5, U_x(0,0) = 0, U_{x^2}(0,0) = U_{x^3}(0,0) = \ldots\ldots\ldots0$

$U_y(0,0) = 5; U_{y^2}(0,0) = U_{y^3} = \ldots\ldots0; \;\; U_{yx}(0,0) = U_{xy(0,0)} = 0; U_{xy^2}(0,0) = -14,$

$U_{xy^3}(0,0) = -42, U_{xy^4}(0,0) = \ldots\ldots\ldots = 0, U_{yx^2}(0,0) = 4; U_{yx^3}(0,0) = \ldots\ldots = 0$

From given conditions

$U(0,0) = 5 = a_0 \ \therefore a_0 = 5; U_x(0,0) = 0, U_{x^2}(0,0) = U_{x^3}(0,0). = \ldots\ldots\ldots 0$

$\therefore b_0 = c_0 = \ldots 0, U_y(0,0) = 5 \ \therefore a_1 = 5$

$$\begin{aligned} U = \ & a_0 + b_0x + c_0x^2 + d_0x^3 + e_0x^4 + \ldots\ldots\ldots\ldots\ldots\ldots \\ & a_1y + b_1xy + c_1x^2y + d_1x^3y + e_1x^4y + \ldots\ldots\ldots\ldots \\ & a_2y^2 + b_2xy^2 + c_2x^2y^2 + d_2x^3y^2 + \ldots\ldots\ldots\ldots\ldots\ldots \\ & a_3y^3 + b_3xy^3 + c_3x^2y^3 + d_3x^3y^3 + \ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$$\begin{aligned} U^2 = \ & {a_0}^2 + 2a_0b_0x + (2a_0c_0 + {b_0}^2)x^2 + (2a_0d_0 + 2b_0c_0)x^3 \\ & + (2a_0e_0 + 2b_0d_0 + {c_0}^2)x^4 + \ldots\ldots\ldots {a_1}^2y^2 + 2a_1b_1y^2x \\ & + (2a_1c_1 + {b_1}^2)y^2x^2 + (2a_1d_1 + 2b_1c_1)y^2x^3 + \ {a_2}^2y^4 \ldots. \\ & + 2a_2b_2y^4x + (2a_2c_2 + {b_2}^2)y^4x^2 + (2a_2d_2 + 2b_2c_2)y^4x^3 + \cdots \end{aligned}$$

$$\begin{aligned} U_y = \ & a_1 + b_1x + c_1x^2 + d_1x^3 + e_1x^4 + \ldots\ldots\ldots\ldots \\ & 2a_2y + 2b_2xy + 2c_2x^2y + 2d_2x^3y + \ldots \\ & 3a_3y^2 + 3b_3xy^2 + 3c_3x^2y^2 + 3d_3x^3y^2 + \ldots \end{aligned}$$

$$\begin{aligned} U_x = \ & b_0 + 2c_0x + 3d_0x^2 + 4e_0x^3 + \ldots\ldots \\ & b_1y + 2c_1xy + 3d_1x^2y + 4e_1x^3y + \ldots\ldots\ldots\ldots \\ & b_2y^2 + 2c_2xy^2 + 3d_2x^2y^2 + .4e_2x^3y^2 \ldots\ldots\ldots\ldots\ldots \\ & b_3y^3 + 2c_3xy^3 + 3d_3x^2y^3 + 4e_3x^3y^3 \ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Comparing with coefficient of	Given $a_0 = 5$, $b_0 = 0$, $c_0 = d_0 = \ldots\ldots\ldots 0$	
Constant term	${a_0}^2 + b_0 + a_1 = 30 \therefore 25 + 0 + a_1 = 30 \therefore a_1 = 5$	$a_1 = 5$
X	$2a_0b_0 + 2c_0 + b_1 = 0 \quad \therefore \quad b_1 = 0$	$b_1 = 0$
x^2	$2a_0c_0 + {b_0}^2 + 3d_0 + c_1 = 2 \therefore \ c_1 = 2$	$c_1 = 2$
x^3	$2a_0d_0 + 2b_0c_0 + 4e_0 + d_1 = 0 \quad \therefore \ d_1 = 0$	$d_1 = 0$
x^4	$2a_0e_0 + 2b_0d_0 + {c_0}^2 + 4e_0 + e_1 = 0 \therefore \quad e_1 = 0$	$e_1 = 0$
y	$b_1 + 2a_2 = 0 \quad \therefore \ a_2 = 0$	$a_2 = 0$
xy	$2c_1 + 2b_2 = -10 \therefore \quad b_2 = -7$	$b_2 = -7$
x^2y	$3d_1 + 2c_2 = 0 \quad \therefore c_2 = 0$	$c_2 = 0$

x^3y	$4e_1 + 2d_2 = 0 \quad \therefore d_2 = 0$	$d_2 = 0$
y^2	$a_1^2 + b_2 + 3a_3 = 0 \therefore \quad a_3 = 7$	$a_3 = 7$
xy^2	$2a_1b_1 + 2c_2 + 3b_3 = -7 \quad \therefore \quad b_3 = -7$	$b_3 = -7$
x^2y^2	$2a_1c_1 + b_1^2 + 3d_2 + 3c_3 = 5 \therefore \; 20 + 3c_3 = 5$ $\therefore c_3 = -5$	$c_3 = -5$

Table: 5.2

Substituting all the values in equation U the solution is

$$U = 5 + 5y + 2x^2y - 7xy^2 + 7y^3 - 7xy^3 + 5x^2y^2 + \ldots\ldots\ldots\ldots\ldots$$

5.12 Partial Derivative and Matrix:

Applications of Partial derivative is in solving determinant of Jacobian Matrix.

Jacobian Matrix:

In Vector calculus or in advance calculus containing multiple variables, Jacobian matrix is written in the form of all partial derivatives w.r.t.one variable, by considering the others as constant.

Let f is a function which is differentiable (all partial derivative exists) at point x,x $\in \mathrm{R}^n$ producef(x) $\in \mathrm{R}^m$.

Then the Jacobian matrix J of function f is of order m × n which is denoted by J_f as follows:

$$J_f = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}_{3\times 3}$$

is an example of 3×3 Jacobian matrix.

If m = n then it is called Jacobian square matrix.

Its determinant is called the Jacobian determinant of x.

It is not necessary that Jacobian matrix is square matrix.

Example: 37

The Jacobian matrix of function f: $R^3 \rightarrow R^3$ is given by$f(x, y) = \begin{bmatrix} x^3y \\ x^2 + y^2 \\ 4x - 3y \end{bmatrix}$

Let $f_1(x, y) = x^3y; f_2(x, y) = x^2 + y^2; f_3(x, y) = 4x - 3y$

$\therefore \frac{\partial f_1}{\partial x} = 3x^2y \therefore \frac{\partial f_1}{\partial y} = x^3$

$\therefore \frac{\partial f_2}{\partial x} = 2x \therefore \frac{\partial f_2}{\partial y} = 2y$

$\therefore \frac{\partial f_3}{\partial x} = 4 \therefore \frac{\partial f_3}{\partial y} = -3$

$$J_f = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix}$$

$J_f = \begin{bmatrix} 3x^2y & x^3 \\ 2x & 2y \\ 4 & -3 \end{bmatrix}_{3\times 2}$ is the Jacobian matrix of order 3x2.

Example: 38

Find determinant of the function with components,

$$f_1(x, y, z) = x\sin y + \cos y + z$$

$$f_2(x, y, z) = \sin x + y - 2z$$

$$f_3(x, y, z) = x^2 + y^2 + z^2$$

$$\frac{\partial \mathrm{f}_1}{\partial \mathrm{x}} = \mathrm{siny}; \frac{\partial \mathrm{f}_1}{\partial \mathrm{y}} = \mathrm{cosy}; \frac{\partial \mathrm{f}_1}{\partial \mathrm{z}} = 1$$

$$\frac{\partial \mathrm{f}_2}{\partial \mathrm{x}} = \mathrm{cosx}; \frac{\partial \mathrm{f}_2}{\partial \mathrm{y}} = 1; \frac{\partial \mathrm{f}_2}{\partial \mathrm{z}} = -2$$

$$\frac{\partial \mathrm{f}_3}{\partial \mathrm{x}} = 2\mathrm{x}; \frac{\partial \mathrm{f}_3}{\partial \mathrm{y}} = 2\mathrm{y}; \frac{\partial \mathrm{f}_3}{\partial \mathrm{z}} = 2\mathrm{z}$$

$$\mathrm{J}_\mathrm{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}_{3\times 3} = \begin{bmatrix} \mathrm{siny} & \mathrm{cosy} & 1 \\ \mathrm{cosx} & 1 & -2 \\ 2\mathrm{x} & 2\mathrm{y} & 2\mathrm{z} \end{bmatrix}_{3\mathrm{x}3}$$

Thus, the Jacobian matrix is

$$\begin{bmatrix} \mathrm{siny} & \mathrm{cosy} & 1 \\ \mathrm{cosx} & 1 & -2 \\ 2\mathrm{x} & 2\mathrm{y} & 2\mathrm{z} \end{bmatrix}$$

The determinant of Jacobian matrix is

$$\begin{vmatrix} \mathrm{siny} & \mathrm{cosy} & 1 \\ \mathrm{cosx} & 1 & -2 \\ 2\mathrm{x} & 2\mathrm{y} & 2\mathrm{z} \end{vmatrix}$$

$$= \mathrm{siny}(2\mathrm{z} + 4\mathrm{y}) - \mathrm{cosy}(2\mathrm{zcosx} + 4\mathrm{x}) + 1(2\mathrm{ycosx} - 2\mathrm{x})$$

Chapter: 6

Applications of Vedic Mathematics in Computer Arithmetic

6.1 Introduction to Computer:

The electronic circuitry within a computer, the Processing Unit the (CPU) of the computer implements instructions given by a computer program by performing basic arithmetic, logical, control and input/output (I/O) operations. These operations are specified by the instructions from which the actual mathematical operation like addition, Multiplication, Division, squaring and cubing for each instruction is performed. The combinational logic circuit responsible for the above function is known as the Arithmetic logic unit or ALU. Mathematical operations find applications in shifting and recovery of distorted streams of information. Squaring & Cubing has the most essential part in Cryptography, Animation, DSP & image Processing etc. where the speed is a crucial performance characteristic. Speed of DSP is a function of speed of Multiplier, adder, Adder- Subtractor, Division and also square and cube architecture. Thus, it is essential to have a faster ALU to meet the needs of this high performance applications.

6.1.1 Multiplier:

The 3 crucial processes of multiplication are creating, reducing and finally adding the Partial product. If a multiplier has to be efficient, it should have features like Accuracy, speedy performance, and energy efficient system & must occupy few number of slices. Based on time taken and area consumed, the construction of multipliers are divided into 3 classes: The Serial multiplier, Parallel Multiplier & Serial Parallel Multiplier. The first stresses on the optimum usage of the hardware specially the chip area. The second does speedy mathematical operations and the third is a transaction between the first two classes.

Different types of Multipliers:

In signed-2's complement Booth multiplier represents a method for multiplying binary integers. The process of multiplication of two unsigned binary numbers as well as multiplication of two signed numbers is done by Combinational multipliers. An effective hardware application of a digital circuit is depicted by the Wallace tree multiplier that multiplies two integers which involves reducing the no. of part products and the addition of the part products is carried out by using carry select adder. The regular structure of Array multiplier it is best suited for the process of repetitive addition and shifting. For low area requirement, Sequential Multiplier is well known.

6.1.2 Adder:

An adder is a digital circuit which can be classified into two types- half Adder and full Adder. The difference between both the type is that by taking sum of 2 digits the first type generates two outputs of a half adder as a carry and sum; whereas the full adder not only adds the sum of the numbers but also keeps record for those values which carried in as well as out. The circuit of 2nd type has varied responsibilities including:

$A + B + C_{in} = C_{out} + S$

Given that input circuit on left side all represent single bit quantities whereas the output on the right is characterized by double bit. S & C stand for Sum and Carry respectively.

The other adders supporting the multiple bits are Carry Save Adder (CSA), Carry look ahead Adder and Ripple Carry Adder (RCA). Among them CSA adds 3 or greater than 3 binary numbers (n-bit) and obtains two outputs of the same dimension as the input in the form of a sequence of carry & partial sum bits; CLA is used to add two binary numbers to reduce the computation time and generates two o/p signals P (Sum-type) & G (Carry-type).

Kogge-Stone adder (KSA) is one of the advanced carry- look ahead architecture. RCA uses multiple full adder for finding the summation of n-bit numbers, in which each full adder depends on the previous one to carry bit for the calculation.

Therefore, RCA is very slow compared to the above two adders.

6.1.3 Types of division algorithm:

In division algorithm, division is of the form N/D = (Q, R) where inputs are N (Numerator) and Divisor D (Denominator) and Outputs are Q (Quotient) and R (Remainder). There are two main categories of division algorithms: Slow & Fast. Every iteration of the slow division algorithm gives 1 digit of the quotient answer, while Fast division algorithm generates twice as many digits of the final quotient on each iteration because it begins with a near estimation of the quotient answer. Examples of slow division are Restore type, Non-Restore type and SRT division. Out of them, Non restore type is better than others. Examples of fast division are Newton-Raphson and Goldschmidt.

6.2 Binary Numbering System and Decimal Numbering System:

The binary system is the internal language of electronic computers.Decimal numbering systems has base 10 i.e. It uses 10 digits from 0 to 9.Binary numbering system has base 2 i.e. it uses only two digits 0 & 1.

6.2.1 Conversion Method of Binary number to Decimal number:

This unique numbering method involves the rise of values from right to left digits by positive integer powers starting from zero on the units place raised on the commom base of two.

By neglecting the corresponding digital value of binary digit value 0 and adding the corresponding digital value of binary digit value 1 the decimal number can be obtained.

Convert binary number 10011101 into decimal number equivalent:

Decimal Digit Value	**2^7 =128**	**2^6 = 64**	**2^5 = 32**	**2^4 =16**	**2^3 = 8**	**2^2 = 4**	**2^1 = 2**	**2^0 =1**
Binary Digit Value	**1**	**0**	**0**	**1**	**1**	**1**	**0**	**1**

Table: 6.1

$$(10011101)_2 = 1 \times 2^7 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^0$$

$$= 1(128) + 1(16) + 1(8) + 1(4)$$

$$= 128 + 16 + 8 + 4 + 1 = 157$$

Thus, $(10011101)_2 = (157)_{10}$

Examples:

$(10)_2 = 1 \times 2^1 + 0 \times 2^0 = 2$, $(100)_2 = 2^2 = 4$, $(1000)_2 = 2^3 = 8$, $(10000)_2 = 2^4 = 16$

$(101)_2 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 4 + 0 + 1 = 5$,

$(1001)_2 = 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 8 + 1 = 9$

$(11111111)_2 = 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

$= 128 + 64 + 32 + 16 + 8 + 4 + 2 + 1 = 255$

6.2.2 Conversion Method of Decimal Number to Binary Number:

We begin by dividing the said decimal number by 2 continuously and write the result. Since it is being divided by 2, for an even dividend -binary remainder will be Zero & for odd dividend binary remainder will be one. Thus a remainder of either 1 or 0 until the final result equals zero. Starting with the bottom remainder, read the sequence of remainders upwards to the top.

Convert Decimal number 158 in binary number system equivalent:

Explanation:

Divisor ÷ Dividend = Quotient + Remainder

$158 \div 2 = 79 + 0$

$79 \div 2 = 39 + 1$

$39 \div 2 = 19 + 1$

$19 \div 2 = 9 + 1$

$9 \div 2 = 4 + 1$

$4 \div 2 = 2 + 0$

$2 \div 2 = 1 + 0$

$1 \div 2 = 0 + 1$

Thus, $(158)_{10} = (10011110)_2$

Table: 6.2

OR

$(158)_{10} = 128 + 16 + 8 + 4 + 2 = 2^7 + 2^4 + 2^3 + 2^2 + 2^1 = (10011110)_2$

For unsigned number to convert decimal into binary is easy. But for signed numbers i.e. negative number it is difficult to convert in binary number. In the computer to convert negative number in binary number negative sign can also be represented as bit. Negative binary numbers are represented by Two's complement notation.

6.2.3 2's complement of Binary No.:

Two's complement is formed as a result of inverting all the bits (i.e. changing each 1 to 0 and each 0 to 1) and then adding 1 to it. The higher order bit of the number (i.e. the leftmost bit) indicates the sign. It indicate positive sign if the leftmost bit is 0 and negative sign if the leftmost bit is 1. The remaining bits represent the magnitude. Thus, using Two's complement all of the arithmetic operations can be performed by the same hardware whether the numbers are signed or unsigned.

Shortcut method to find the 2's complement:

Starting from the right i.e. least significant digit to the left i.e. most significant digit, select digit 1 which first time.

Reverse (i.e. change 0 to 1 and 1 to 0) all of the bits to the left of that '1'

E.g.

Find 2's complement of 11010100.

Starting from the right 3rd digit is first 1, so invert all the bits after that 1.

Thus, 2's complement of 11010100 is 00101100

OR

Invert all the bits and then add 1 to it. i. e. 00101011 + 1 = 00101100

6.3 Arithmetic operations in Binary Numbering System by Vedic Sūtras:

In binary arithmetic operations; Addition, Subtraction, Multiplication and Division are performed by using two bits 0 and 1.

By taking one example of each operation it is explained by the researcher.

6.3.1

Binary Addition:

11100011 + 01111011 = 101011110

  11100011
+ 01111011
 101011110

6.3.2

Binary Subtraction:

11111000 – 01010100 = 10100100

$$\begin{array}{r} 11111000 \\ -\ \underline{01010100} \\ 10100100 \end{array}$$

6.3.3

Binary Multiplication:

111001 x 101011

$$\begin{array}{r} 111001 \\ \times\ \ \underline{101011} \\ 111001 \\ 111001 \\ 000000 \\ 111001 \\ 000000 \\ 111001 \\ \ldots\ldots\ldots\ldots\ldots \\ 1\,1\,0\,1\,0\,0\,1\,1 \end{array}$$

6.3.4

Binary Division:

111111 ÷1001 = 111

$$\begin{array}{r|l} & 111 \\ \cline{2-2} 1001 & 111111 \\ & \underline{1001} \\ & 01101 \\ & \underline{1001} \\ & 01001 \\ & \underline{1001} \end{array}$$

0000

Thus, 111111 ÷ 1001 then Quotient Q = 111 and Remainder R = 0

In Octal base is 8 i.e.it uses 8 digits 0,1,2,3,4,5,6, 7 and 8 becomes 10.

$(10)_8 = 1 \text{ x } 8^1 + 0 \text{ x } 8^0 = 8.$

In hexadecimal base is 16 i.e. it contains digits 0, 1, 2,3,5,6,7,8,9

A = 10, B = 11, C = 12, D = 13, E = 14 and F = 15.

Vedic Sūtras: computer applications:

In most DSP processor multiplier is one important part of hardware because of that speed of processor increases. Finite Impulse Response filters normally called as a convolution filter (FIR and IIR Filter) includes a multiplier in it. Execution of FFT needs huge no. of multiplications & additions and because of that it becomes very complicated. So it is crucial to have high speed and power consuming multiplier to make it simple and rapid.

In RSA cryptography fix and floating point division is performed. Division is very time consuming process. So there is a need for efficient division architecture to achieve high speed cryptography algorithm in secure transaction.

In Elliptic Curve Cryptography point additions and doubling as exponential operations like squaring, cubing and 4th power occur in these operations as they are more time intensive arithmetic processes. So there is a need to have time, Area and Power efficient Square and Cube architecture.

To overcome the above needs Vedic Mathematics **Sūtras** are very helpful.

Vedic Mathematics **Sūtras** are applied in decimal number system as well as binary number system also.

In this chapter **Vertically & Crosswise Sūtra** based multiplier for all types of multiplication, **Nikhila□ Sūtra** based multiplier for the special type of multiplication, **Ekanyūnena Pūrve□a** Vedic Multiplier and **Ānurūpye□a Sub-Sūtra** based multiplier for special type of multiplication increase the modularity while reducing complexity of the design required to input bigger bit numbers. Binary division by Nikhila□ method for the numbers which are near to base 10 further more simplifying the problem and reducing the number of elements, Parāvartya Yojayet Sūtra as against Nikhila□ Sūtra for helping in solving bigger problems of binary division, Dhwaja□ka a generic method applicable to all type of binary division, squaring by **Dwandwayoga** means Duplex for a multiplication of number by itself (i.e. for squaring) & also Yāvadūnam Tāvadunī Kr□tya Vargañaca Yojayet Sub-Sūtra for squaring and cubing architecture and Ānurūpye□a Sub-Sūtra for cubing are discussed by the researcher.

Vedic Multiplier, Vedic Squaring and Cubing architecture are the most efficient multiplier output relating to space, less power consumption & rapidity than the conventional multipliers like Sequential, Booth, Array, Wallace Tree and Combinational multipliers.

6.4 Binary Multiplication by using Vedic Sūtra:

6.4.1 By using Vertically & Crosswise Sūtra based Vedic Multiplier:

In DSP Vedic Mathematics algorithm is applied to digital multiplier. Ūrdhva-Tiryagbhyām algorithm applied to decimal number system is also applied to binary number system. In binary system base is two. i. e. only 0 and 1 (two bits) are used, therefore multiplication by using Vertically & Crosswise Algorithm can be exchanged by AND logic where every AND would be a bit broad. To generate cross-product these bits are added together.

Algorithm:

Single bit vertical product is carried out between bits of minimum significance (LSBs) of multiplicand and multiplier.

Further calculation is done with 2-bits crosswise product and by incrementally adding by 1 bit until all bits are exhausted.

Moreover decreasing bits from LSB for cross-multiplication and continuing this until MSB (only) is useful for vertical multiplication.

Concatenate all the result by carry over extra bit which is the final answer.

Architecture of Vedic multiplier by Vertically & Crosswise Sūtra:

Vertically & Crosswise Sūtra based Vedic multiplier is applicable for multiplication and it reveals the efficiency of reducing the N × N bit multiplier to 2 × 2 bit structure.

The N × N bit multiplier structure is executed by N × N gates and N half adders and (N-2)*N Full adder i.e. total (N-1) * N adders.

The 2 × 2 multiplier is executed by 4 input AND gates and 2 half adders.

The 4 × 4 multiplier is executed by 16 input AND gates 4 half adders and 8 full adders.

An 8 × 8 - bit multiplier is executed by Four 4×4 -bit multipliers and Three 16 × 16 bit adders.

With the help of the basic 2 × 2 bit multiplier, multiplier with 4 by 4 bit developed, with the help of 4 × 4 bit block, 8 by 8 bit block, 16 by 16 bit multiplier and then finally 32 by 32 bit multiplier based on Vedic Sūtra has been designed and thus ALU design with Vedic overlay algorithm is efficient related to rapidity, area and power consumption.

Binary Multiplication of 2 x 2 bit:

(By using Vertically & Crosswise Sūtra based Vedic Multiplier)

Let $A = A_1 A_0$ and $B = B_1 B_0$ are given two bit nos.

For finding $A \times B$

$$\begin{array}{lll} & A_1 & A_0 \\ \times & B_1 & B_0 \\ \hline A_1B_1 \mid & A_0B_1 + A_1B_0 \mid & A_0B_0 \\ C_1 & & \\ C_2S_2 \mid & C_1S_1 & \mid S_0 \\ C_2 \quad S_2 & S_1 & S_0 \end{array}$$

Thus, $A_1A_0 \times B_1B_0 = C_2S_2S_1S_0$

It can be explained by figure using **Vertically & Crosswise Sūtra**:

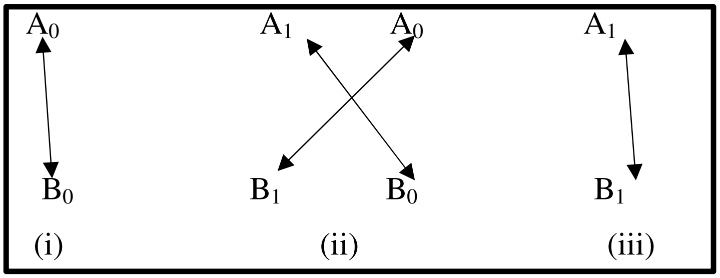


Figure: 6.1

Algorithm:

First, start with 1 x 1 bit vertically multiplication of the multiplicand (A_0) with the least significant Bit of multiplier (B_0).i.e. $A \times B_0 = S_0$ which is the LSB of the answer.

Find crosswise multiplication of 2 × 2 bit of LSB of multiplier with next higher bit of the multiplicand and evaluate another product similarly by interchanging digits in vice versa format. Then add both the product and the obtained result gives the second bit of the final answer.

For 2 × 2 bit maximum cross product width is 2. i.e. $A_0B_1 + A_1B_0 = C_1S_1$ where the S_1 is the second bit of the final answer and carry C_1 add to the next step.

Now 1 × 1 bit vertically product of the most significant bit (MSB) of multiplicand (A_1) with MSB of multiplier (B_1) and then add the carry C_1 to that product.

i.e. $A_1 \times B_1 + C_1 = C_2S_2$. Here sum S_2 is the third bit of the final answer and carry C_2 becomes the most significant bit of the final answer.

Thus the final answer is $A_1A_0 \times B_1B_0 = C_2S_2S_1S_0$.

Binary multiplication of 4x4 bit by using Vedic Multiplier:

1111

× 1010

1|01|10|010|01|01|0

1| 1|01| 0 | 0| carry

---

10 0 1 0 1 1 0

Thus, 1111 × 1010 = 10010110

Explanation:

Starting multiplication either from LSB or MSB with 1 × 1 bit, then continuing 2 × 2, 3 × 3 and till 4 × 4 and again decreasing order of bit 3 × 3, 2 × 2 and 1 × 1 bit.

1111×1010

1	11	111	1111	111	11	1
1	10	101	1010	010	10	0
1	01	10	010	01	01	0

$$\begin{array}{ccccccccl}
1 & 1 & 0 & 0 & 1 & 1 & 0 & & \\
0 & 1 & 1 & 0 & 0 & & & & \text{carry} \\ \hline
10 & 0 & 1 & 0 & 1 & 1 & 0 & &
\end{array}$$

Thus, 1111 × 1010 = 10010110

Figure: 6.2

In N × N bit multiplication maximum cross product width will be $\log_2 N + 1$.

In 8 × 8 bit multiplication maximum cross product width will be $\log_2 8 + 1 = 4$ and for 4 × 4 it will be $\log_2 4 + 1 = 3$ for 2 × 2 it will be $\log_2 2 + 1 = 2$.

There is alternative way for finding multiplication by using Vertically & Crosswise Sūtra which is explained by the following figure.

An alternative way of 4 x 4 bit multiplication: Calculate: 1111 ×1010

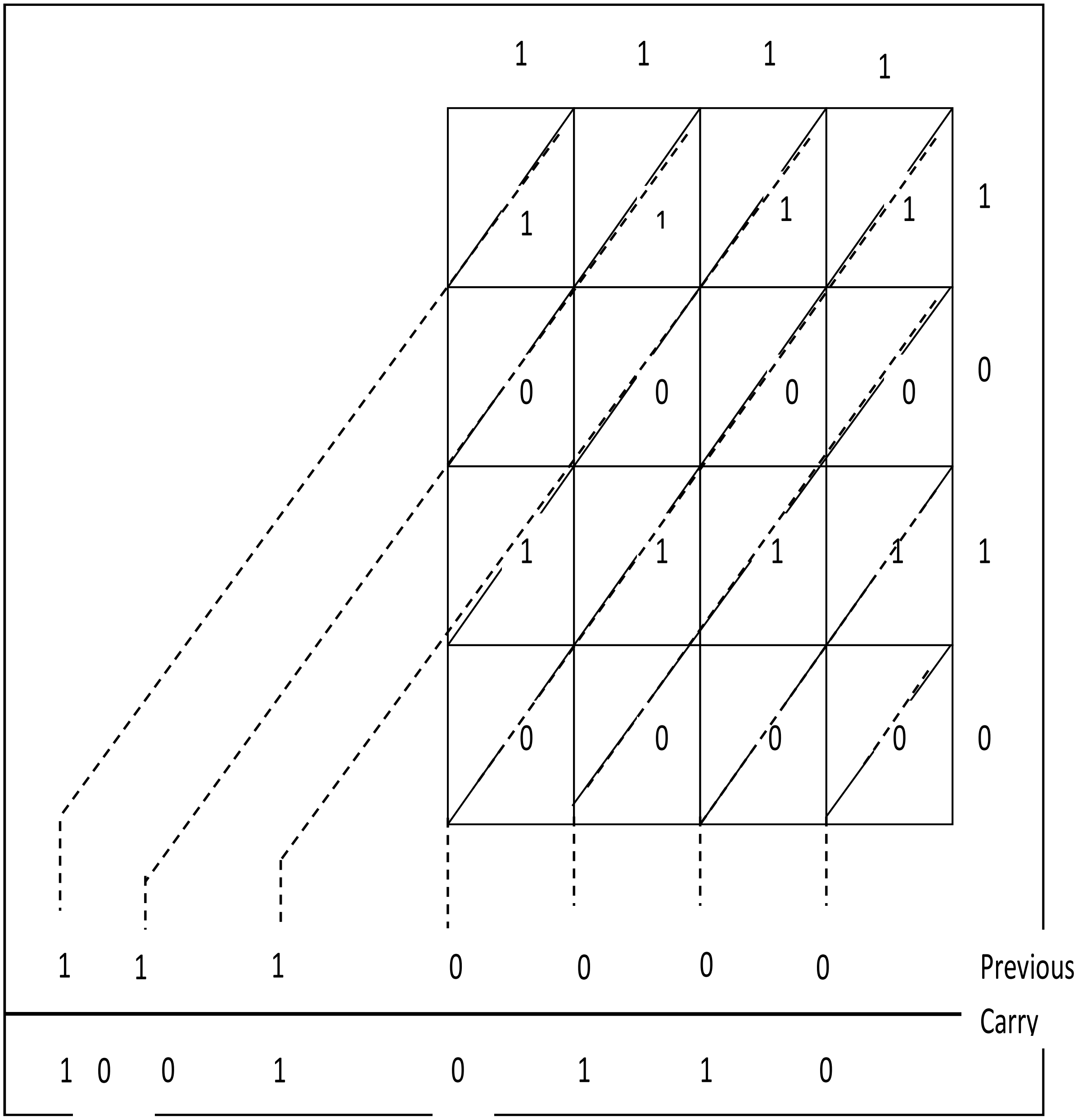


Figure: 6.3

Thus, 1111 × 1010 = 10010110

As the figure above bits of multiplier and bits of multiplicand are written to the consecutive sides of a square.

The horizontal and vertical arrangements of numbers carry information about the bits of quantities whose product is to be found.

These multiplier bits are multiplied with each bits of multiplicand independently and write the product in the common box.

Initially, carry for RHS start is assumed 0 as per the standard format.

Addition is done diagonally along the slant path highlighted along with an additional entity called the preceding carry-over from neighbour.

From obtained product, the part of the final answer possessing minimum significance is retained whereas the remaining is passed to next neighbour as carry-over and this operation continues till it reaches the bit of greatest significance & thus, the desired product is obtained.

Binary multiplication of 8x8 bit by using Vedic multiplier:

Calculate: 10111111 × 11000011

1 0 1 1 1 1 1 1

× 1 1 0 0 0 0 1 1

1| 01|01|010|010|010|011|0011|010|010|010|010|10|10|1

1| 1|10 |10 | 10 |10 | 10| 1 | 1 | 1 | 1 | 1 | 1 Carry

10 0 1 0 0 0 1 0 1 1 1 1 1 01

Thus, 10111111 ×11000011 = 1001000101111101

Explanation:

10111111 × 11000011

1	10	101	1011	10111	101111	1011111	10111111
1	11	110	1100	11000	110000	1100001	11000011
1	01	01	010	010	010	011	0011

0111111	111111	11111	1111	111	11	1
1000011	000011	00011	0011	011	11	1
010	010	010	010	10	10	1

1 1 1 1 1 1 1 carry

1 1 1 0 0 0 1 1 0 0 0 0 0 0 1

1 1 1 1 1 1 1 1 1 1 1 carry

1 0 0 1 0 0 0 1 0 1 1 1 1 1 0 1

Thus, 10111111 ×11000011 = 1001000101111101

Figure: 6.4

Example of 8 × 8 bit Binary Multiplication:

Calculate: 11111111 × 11111111

1 1 1 1 1 1 1 1

1 1 1 1 1 1 1 1

………………………………..……………………………………………..

1 | 10 | 11 |100 | 101 | 110 | 111 | 1000 | 111 | 110 | 101 | 100 |11 | 10 | 1

10 11 100 101 110 111 111 110 101 100 11 10 1

……………………………………………………………………………………

1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 1

Thus, 11111111 × 11111111 = 1111111000000001

16×16 bit and 32×32 bit multiplication can also be done by using Vedic multiplier.

6.4.2 Binary multiplication by using Nikhilam□ Sūtra based Multiplier:

Nikhila□ Sūtra based multiplier is more efficient in the multiplication of large numbers as it reduces the multiplication of two large numbers to that of smaller.

The above Sūtra can also be applicable to binary number system. In binary system deviation from base can also be found by taking 2's complement of that number. For an eight bit Vedic Multiplier, the R.H.S. part of the product is implemented using 8- bit carry save adder. Hence multiplication of two 8-bit numbers is reduced to the multiplication of their compliments and addition. Nikhila□ Sūtra is basically more efficient when both the multiplier and multiplicand are near to same base power. For binary numbers base is in the power of 2.

e.g. $2^1 = 2 = (10)_2$, $2^2 = 4 = (100)_2$, $2^3 = 8 = (1000)_2$

Algorithm:

First find the nearest same base bits of the given multiplier and multiplicand both.

Then find deficit bit by subtracting the base bit from both multiplier and multiplicand.

Obtain the adjusted insufficiency and place these results across corresponding row of multiplier and multiplicand.

Multiply that two deficit bits which is the R. H.S. of the final answer.

Then either add the deficit of multiplicand to the given multiplier or add the deficit of multiplier to given multiplicand. The obtained result is the L.H.S. of the final answer.

2×2 bit:

Calculate: 11 × 11

Here base is 10

11 (11-10) =1
× 11 (11-10) =1
(11 + 1) | (1×1)
100 | 1
Thus, 11 × 10 = 1001

3×3 bit:

Calculate: 101 × 110

Here base is 100

101 (101-100) = 1
× 110 (110-100) = 10
(110 + 1) or (101 + 10) | (1×10)
111 | 10
Thus, 101 × 110 = 11110

4× 4 bit:

Calculate: 1111 × 1111

Here base is 1000

1111 (1111-1000) = 111
× 1111 (1111-1000) = 111
(1111+111) | (111 × 111)
10110 | (111 ×111)
10110 | 110001 (111 × 111 = 110001)

110 | 001 Carry (base 1000)

111100 | 001

11100001

Thus, 1111 × 1111 = 11100001

Explanation of 111 × 111 used in above example:

111 ×111(Base is 100)

111 (111-100) = 11

× 111 (111-100) = 11

(111+11) | (11×11)

1010 | (11 ×11)

1010 | (1001)

1010 | 01

10 Carry (base 100)

1100|01 Thus 111 × 111 = 110001

Explanation of 11×11 used in above example:

11 × 11 (base 10)

11 (11-10) = 1

× 11 (11-10) = 1

(11 + 1) | (1 × 1)

100 | 1 Thus, we get 11 × 11 = 1001

Explanation by alternative way:

Calculation for 4 x 4 bit	Base difference	Next difference	Next difference

1111 × 1111 (1111 + 111) × 1000 + (111 × 111) = 10110 × 1000 +110001 = 10110000 +110001 = 11100001 Thus, 1111 × 1111 = 11100001	(1111 -1000) = 111 (111 + 11) × 100 + (11×11) =1010 × 100 + 1001 = 101000 + 1001 = 110001 Thus, 111 × 111 = 110001	111 – 100 = 11 (11 + 1) × 10 + 1×1 = 100 × 10 + 1 = 1000 + 1 = 1001 Thus, 11 × 11 = 1001	11 – 10 = 1 1 × 1 = 1

Table: 6.3

In the same way 11111111 × 11111111(8 × 8 bit) can be solved as below in table: 6.4

Next diff.	Next diff.	Next diff.	Next diff	Next difference
111111 -100000 11111 ×(11111 +1111) × 10000 + 1111 x 1111 = 101110 × 10000 +11100001 =1011100000 + 11100001 =1111000001 11111 x 11111 = 1111000001	11111 -10000 1111 (1111+111)10 00 + 111 x 111 =101110 ×1000 +11001 =101110000 + 11001 =11100001 1111 x 1111 = 11100001	1111-1000 =111 (111+11)10 0 +11 x 11 =1010 × 100 + 1001 = 101000 + 1001 =110001 111 x 111 =110001	111-100 =11 (11 + 1)10 + 1 x 1 = 100 (10) + 1 = 1000 + 1 = 1001 11 x 11 1001	11-10 = 1 1 x 1 = 1

Calculation for 8x 8 bit	Base difference	Next difference
11111111	11111111	1111111
x11111111	-10000000	-1000000
	1111111	111111
(111111111+1111111)	(1111111 + 111111)	(111111 + 11111)
×10000000	×1000000	100000
+1111111 x 1111111	+111111 x 111111	+ 11111 x 11111
=101111110	= 10111110	= 1011110
× 10000000	× 1000000	×100000
+11111100000001	+111110000001	+1111000001
=1011111100000000	= 10111110000000	=101111000000
+ 11111100000001	+ 111110000001	+ 1111000001
=1111111000000001	=11111100000001	=111110000001
11111111 x 11111111	1111111 x 1111111	111111 x 111111
= 1111111000000001	= 11111100000001	= 111110000001

Table: 6.4

With rise in technological advances and research in material science, future trends predict complete eradication of outdated irreversible operations of logic elements which thrive on deleting valuable information during processing. The inherent characteristic of these futuristic inventions will rely on the Nikhila□ Sūtra algorithm helping in fast and effective computational performance.

6.4.3 Binary multiplication by Ānurūpye□a Sub-Sūtra based Multiplier:

When both the numbers are near the same base but other than the theoretical base like 10,100, 1000, 10000…… and if we apply Nikhila□ Sūtra then after subtracting the given multiplier and multiplicand from the base we get the result very big number. By using **Ānurūpye□a Sub-Sūtra** means proportionately find the working base and compare it with theoretical base and find the proportion between the two bases. In the final answer only left part is proportionately multiply.

By Nikhilam Method

1101 × 1100

By Nikhilaṃ nearest base is 1000

1101 (1101 - 1000) = 101
×1100 (1100 - 1000) = 100
(1101+100) or (1100 +101) | 101 × 100
10001| 10100
10001+10 | 100
10011100 **Thus, 1101 × 1100 = 10011100**

By Anurūpyeṇa Method

By using Anurūpyeṇa, theoretical base is 100

Working base is 1100 = 11 ×100

1101 (1101-1100) = 01
× 1100 (1100 -1100) = 00
(1100 + 01)(1100 + 00) | 00
1101 | 00
1101 × 11 | 00
10011100 **Thus, 1101 × 1100 = 10011100**

6.4.4 Binary multiplication by Ekanyūnena Pūrveṇa Sūtra based multiplier:

Ekanyūnena Pūrveṇa Sub-Sūtra means, “one less than the previous one”. The greatest benefit is that this sūtra comes handy during hard calculations involving figures of type 9, 999, 999, 9999 & so on.

Avoiding unnecessary complexity significantly reduce the time delay and makes the system faster.

In Binary system base is 2. This Sūtra is useful for multiplication of numbers with the multiplier which are in the form of $2^n – 1$.

i.e. Multiplication of binary number with multiplier like 1, 11, 111, 1111, 11111……..

First subtract 1 from the multiplicand i.e. given binary number other than multiplier. The obtained result is the left part of the final answer.

The obtained result subtract from the multiplier (which is in the form of 1, 11, 111, 1111…….). That result is the right part of the final answer.

E.g.

Calculate: 1111 × 1000

(1000- 1) = 0111, which is the left part of the final answer.

(1111- 0111) = 1000, which is the right part of the final answer.

1 1 1 1
× 1 0 0 0
0111 | 1000

The final answer 1111×1000 = 01111000.

Calculate: 10101 ×11111

(10101 – 1) = 10100

(11111- 10100) = 01011

10101
×11111
10100 | 01011

Thus, 10101 × 11111 = 1010001011

6.5 Binary Division by using Vedic Sūtras:

6.5.1 Binary Division Algorithm by using Nikhila□ Sūtra:

Let A be the dividend (numerator), B is the divisor (denominator) and N represents number of bits of the divisor. First find the deficit bits by subtracting 10N from the divisor.Separate the dividend bits into two parts one for the quotient bits and another for remainder bits such that Remainder bits is equal to the divisor bits and the number of dividend bits – number of divisor bits = number of quotient bits.

Write the dividend bits and remainder bits in the first row and write first bit of the dividend part under the first bit below the last row in the first column which is the first bit Q_1 of the final answer of the quotient. Then find product of deficit bits D_1 D_2 and Q_1 which is P_1 P_2 write it below the second and third bits of dividend.

Add the second bit of dividend to P_1 and write it below the last row as Q_2.

Till the last bit continue this process. The last quotient bit considered for finding remainder R_1, R_2…….

Example of binary division by Nikhila□ Sūtra:

[1] 1100110 ÷ 1001

Nearest base of 1001 = 10000, Deficit bits = 10000 -1001 = 0111

OR Deficit Bits = 2's complement of 1001 = 0111

Divisor bits **1 0 0 1**	Dividend bits **1 1 0 0 1 1 0**	
Deficit Bits: **0 1 1 1**	Bits allotted for finding quotient bits	Bits allotted for finding remainder bits
	1 1 0	**0 1 1 0**
	0 1	1 1
	0	1 1 1
		0 1 1 1
	1 1 1	**10 100 11 1**
		10 1
	1 1 1	**100 1 1 1**
Therefore, Q = 1 1 1 R = 100 111		

Table: 6.5

Here, Remainder is greater than Divisor. So considering Remainder as dividend divide it by the divisor same process as above.

Divisor bits **1 0 0 1**	Dividend bits = Previous Remainder **1 0 0 1 1 1**	
Deficit Bits:	Bits allotted for finding	Bits allotted for finding

0 1 1 1	quotient bits	remainder bits
	1 0	**0 1 1 1**
	0	1 1 1
		0 0 0 0
	1 0	**1 10 10 1**
		1 1
	1 0	**10 1 0 1**
Therefore, Q = 111+ 10 =1101 R = 10101		

Table: 6.6

Remainder is also greater than the divisor. Repeat the above procedure.

Divisor bits **1 0 0 1**	Dividend bits = Previous Remainder **1 0 1 0 1**	
Deficit Bits: **0 1 1 1**	Bits allotted for finding quotient bits	Bits allotted for finding remainder bits
	1	**0 1 0 1**
		0 1 1 1
	1	**0 10 1 10**
		1 1 1
	1	**1 1 0 0**
Therefore, Q = 111+ 1 0 +1 = 1010 R = 1100		

Table: 6.7

Here also remainder is greater than divisor .Considering the remainder as dividend repeat the procedure till remainder is either equal or less than the divisor.

Divisor bits	Dividend bits = Previous Remainder
1 0 0 1	1 **1 0 0**

Deficit Bits: **0 1 1 1**	Bits allotted for finding quotient bits	Bits allotted for finding remainder bits
	0	**1 1 0 0**
		0 1 1 1
	0	**1 10 1 1**
		1
	0	**1 0 0 1 1**
	1	**0 0 1 1**
Therefore, Final quotient Q = 111+10 +1+1 = 1011 , Final R = 0 0 1 1		

Table: 6.8

Here, remainder is less than divisor so Final answer is Q = 1011 and R = 0011

[2] 1101011 ÷ 1101

Divisor its **1 1 0 1**	Dividend bits 1 1 0 1 0 1 1	
Deficit Bits: **0 0 1 1**	Bits allotted for finding quotient bits	Bits allotted for finding remainder bits
	1 1 0	**1 0 1 1**
	0 0	1 1
	0	0 1 1
		0 0 0 0
	1 1 0	**10 10 10 1**
		1 1
	1 1 0	**11 1 0 1**
Therefore , Q = 110 and Remainder = 11101		

Table: 6.9

Here remainder is greater than the divisor. Considering the obtained remainder as dividend continuing the above procedure.

Divisor bits **1 1 0 1**	Dividend bits = previous remainder **1 1 1 0 1**	
Deficit Bits: **0 0 1 1**	Bits allotted for finding quotient bits	Bits allotted for finding remainder bits
	1	**1 1 0 1**
		0 0 1 1
	1	**1 1 1 10**
		1 1 1
	1	**10 0 0 0**
Therefore, Q = 110 + 1 = 111 & R = 10000		

Table: 6.10

Remainder is greater than the divisor. So considering the remainder as divisor continuing the same procedure as above.

Divisor bits **1 1 0 1**	Dividend bits **1 0 0 0 0**	
Deficit Bits: **0 0 1 1**	Bits allotted for finding quotient bits	Bits allotted for finding remainder bits
	1	**0 0 0 0**
		0 0 1 1
	1	**0 0 1 1**
Therefore , Final Quotient Q = 111 + 1 = 1000 and Final R = 0 0 1 1		

Table: 6.11

6.5.2 Binary Division by Parāyartya Yojayet Sūtra:

Process of Parāvartya division is similar to the Nikhila□ method but here the Deficit bits are replaced with transposed bits of denominator bits. Parāvartya Yojayet Sūtra means transpose and Apply.

Take the transpose of the bits of the divisor by keeping first bit of divisor as it is and change the sign of the remaining bits of the divisor and dividing the dividend bits by that transposed bits, Quotient and Remainder is obtained.

Examples:

[1] 1100110 ÷ 1001

Divisor bits **1 0 0 1**	Dividend bits 1 1 0 0 1 1 0	
1 0 0 1 0 0 -1	Bits allotted for finding quotient bits	Bits allotted for finding remainder bits
	1 1 0 0	**1 1 0**
	0 0 -1	
	0 0	-1
	0	0 0
		0 0 1
	1 1 0 -1	**0 1 1**
Therefore, Quotient Q = 1000 + 100 - 1 = 1101 and R = 0 1 1		

Table: 6.12

[2] 1101011 ÷ 1101

Divisor bits **1 1 0 1**	Dividend bits 1 1 0 1 0 1 1	
1 1 0 1 -1 0 -1	Bits allotted for finding quotient bits	Bits allotted for finding remainder bits
	1 1 0 1	**0 1 1**
	-1 0 -1	
	0 0	0
	0	0 0
		0 0 0

	1 0 0 0	**0 1 1**
Therefore,	**Quotient Q = 10000 and**	**R = 011**

Table: 6.1

6.5.3 Binary Division or Direct Flag Division by using Dhwaja□ka property of Vertically & crosswise Sūtra:

Even though conventional division works fine for low to moderate range numbers, the traditional mehod becomes notoriously difficult when dealing with extreme figures. In such cases, Dhwaja□ka feature of VM can come to our rescue working as a short cut to lengthy calculations. In this process, division is carried out in parts & thus, a big clumsy operation reduces to several small easier ones. This splitting is carried out among all parameters of remainder, divisor, quotient and dividend.

Divison takes place bitwise in a specific order & according to standard guidelines. Remaining bits of the divisor are defined as flag type. Since we are not engaging with the entire combination of bits at one time but flagging after specific intervals, the overall effort required drastically drops down.

Intermediate dividends like Flag (FD), Gross (GD), Net (ND) are evaluated which in turn are employed to find the desired solution in terms of Quotient & Remainder.

Mean computation period for this novel method can be verified to be relatively much lower than regular technique especially for larger quantities.

Also for binary numbers the division method is similar to the division process for decimal numbers,

Examples:

[1] 11011101 ÷ 1101

Divisor bits **1 1 0 1**	Dividend bits **1 1 0 1 1 0 1**	
1 0 1	Bits allotted for finding quotient bits	Bits allotted for finding remainder bits
1	**1 1 $_0$0 $_0$1**	**$_0$1 0 1**

	1 0 0 0	**1 0 1**
Therefore, Quotient Q = 1000 and R = 1 0 1		

Table 6.14

Remainder = 101 – [100(0) + 10 (0) + 0] = 101 – 0 = 101

[2] 1100110 ÷ 1001

Divisor bits **1 0 0 1**	Dividend bits **1 1 0 0 1 1 0**	
0 0 1	Bits allotted for finding quotient bits	Bits allotted for finding remainder bits
1	**1** $_{0}$**1** $_{0}$**0** $_{0}$**1**	$_{0}$**1 1 0**
	$_{1}$**0**	
	1 1 0 0	
Therefore, Quotient Q = 1 1 0 0 and R = 0 1 0		

Table: 6.15

Remainder = 110 - [100 (1) + 10 (0) + 0] = 110 -100 = 010

6.6 Binary Squaring:

6.6.1 Algorithm for finding Binary Square by using Yāvadūnam Tāvadūnīkṛtya Varganca yojayet Sub-Sūtra:

The Yāvadūnam Sūtra with the same procedure as in decimal system is applied for the binary number system (Base = 2^n) in which for finding squares. Algorithm for finding square is as follows.

As the base is always greater than n-bit number, find this deficiency. Square of that deficiency is the R.H.S. of the final answer.

This methodology of defecit estimation resembles the steps involved in twos complement calculation.

The L.H.S. of the final answer is obtained by subtracting the deficiency of the given n-bit number (which is already found in step-1) from the given n-bit number of which we want to find out the square. The number of bit in the R.H.S. of the final answer is depend on the base. E.g. If base is 2^2 (i.e. 4 = 100) then R.H.S. is of 2 bits.

The subtraction operation of the deficit of the n-bit number can also be competent by a left shift operation. This can be explained as left shift by a single bit ignoring the

(n-1)th bit and assigning the value '0' to LSB.

By taking an example the algorithm of square architecture by using Yāvadūnam Sūtra can be explained:

E.g. Calculate $(1101)^2$

Algorithm of Square Architecture by using Yāvadūnam:

Considering a 4-bit binary number say $(1101)_2 = (13)_{10}$, then the nearest base of $(13)_{10}$ in the form of power-2 is $2^4 = (16)_{10}$. i.e. $2^4 = (16)_{10} = (10000)_2$

Step-1: 10000-1101 = 0011 which is the deficit from the nearer base which is exactly equal to compement of two.

Step-2: The square of Deficit i.e. $(1101)^2 = 1001$ which is the R.H.S. of the final answer.

Step-3: Subtract the deficit from 1101 i.e. 1101-0011=1010 which is the L.H.S. of the final answer. This can be explained as left shift operation of the given 4-bit binary number 1101. By ignoring the (4-1)th bit i.e. 3rd bit and assigning LSB to 0 we get the result as 1010 which is exactly same as the output what we obtained by subtracting the deficit 0011 from the 4-bit number 1101.

Step-4: Concatenate the L.H.S. and the R.H.S. and by carry over the extra digit starting from the right side the final answer can be obtained.

Thus, $(1101)^2 = 10101001$

The Yāvadūnam Sūtra based squaring architecture is time and area efficient than the conventional method of squaring. So it may be used in computer graphics, Cryptography and implementation in ALU circuits.

6.6.2 Duplex property of Ūrdhva- Tiryagbhyām Sūtra:

First of all, the figure in the problem is analyzed & identified whether the string length is of even nature.Next, pairing is done from the outward to inward direction.These newly formed pairs are multiplied by two and the results are noted. In case of odd string length, the unpaired central term is squared. These individual results are combined to evaluate the value of final solution.

Algorithm:

For n-bit number starting from MSB to find the duplex with 1 bit.

Then find duplex by decreasing 1 bit continuously till we reach the LSB with 1-bit.Concatenate all the result to get the final answer.

Squaring is done by using Vertically & Crosswise Sūtra by multiplication of given binary number with itself.

Comparing the answers of the Vertically & Crosswise Sūtra starting with the Least Significant Bit of each steps one by one with the answers of Duplex property starting with1-bit, it can be found that both the answers are same.

Therefore, the duplex property is known as the duplex property of Vertically & Crosswise Sūtra.

Algorithm for Binary number for calculating $(1101)^2$:

Duplex Property:

[1] D (1) = $1^2 = 1$
[2] D (11) = 2*1*1=2 = $(10)_2$
[3] D (110) = 2*1*0 +1^2 = 1
[4] D (1101) = 2*1*1 +2*1 *0 =2+0 = 2 = $(10)_2$
[5] D (101) = 2*1*1 + 0^2 =2= $(10)_2$
[6] D (01) = 2*1*0 = 0
[7] D (1) = $1^2 = 1$

$(1101)^2$ = D (1)| D (11)| D (110)| D (1101)| D (101)| D (01) | D (1)
1 | 10 | 1 | 10 | 10 | 0 | 1
1 | 1 | 1 | 1 carry

1 0 1 0 1 0 0 1

Thus, $(1101)^2$ = 10101001

Figure: 6. 5

By using Ūrdhva - Tiryagbhyām Sūtra:

$$\begin{array}{llllllll}
1 & & 1 & & 0 & & 1 & \\
\times\ 1 & & 1 & & 0 & & 1 & \\ \hline
1 \mid & 10 \mid & 1 \mid & 10 \mid & 10 \mid & 0 \mid & 1 & \\
1 & 1 & 1 & 1 & & & & \text{carry} \\ \hline
10 & 1 & 0 & 1 & 0 & 0 & 1 &
\end{array}$$

Thus, $(1101)^2 = 10101001$

A novel technique of sūtra squaring has been devised to compliment the pros of speed algorithms and robustness of traditional multipliers

Similarly, Binary Square of 8-bit, 16-bit and 32-bit can be find out by the same procedure.

6.7 Binary cubing:

6.7.1 Yāvadūnam Sūtra based cubic algorithm:

Just like the square of a number cube of a number can be find out by using Yāvadūnam

Algorithm for finding cube of a binary number:

First find the deficit of the number double the deficit and add to the original number. New number can be obtained which is the L.H.S. of the final answer.

I.e. New number = original number + (2 × deficit)

Find new deficit from the new number obtained in the above step

And multiply new deficit and deficit of the original number which is the middle part of the final answer.

Take the cube of the old deficit which is the R.H.S. of the final answer.

Thus we get,

$$(\text{Original number})^3 = \text{original number} + (2 \times \text{deficit}) \mid (\text{original deficit} \times \text{new deficit}) \mid (\text{original deficit})^3$$

Concatenate all the three parts final answer can be obtained.

E.g: $(101)^3$

Nearest base is Original deficit = 101 - 100 = 001

New number = 101 + 10 × (001) = 111

New deficit = 111 - 100 = 011

New Deficit × Original deficit = 1 x 011 = 011

Cube of original deficit = $(1)^3$ = 01

Thus,

$(101)^3$ = 110 | 11 | 01

$\underline{\qquad\qquad 1| \quad \text{Carry}}$

111|11| 01

The final answer = 1111101

6.7.2 Ānurūpyeṇa cubic algorithm:

Algebraic identity:

$(a + b)^3 = (a^3 + 3a^2b + 3ab^2 + b^3)$

$$\begin{array}{cccc} a^3 & a^2b & ab^2 & b^3 \\ & 2(a^2b) & 2(ab^2) & \\ \hline a^3 & 3a^2b & 3(ab^2) & b^3 \end{array}$$

(Double the middle two terms of the first row)

Algorithm for finding cube of a binary number by Anurūpyeṇa:

Let n-bit binary number M is given. First divide the given number in two partition of n/2 bits which is A and B.

Find the cube of A (M.S.B.) which is L.H.S. of the final answer and the cube of B (L.S.B.) which is the R.H.S. part of the final answer.

For finding intermediate terms calculate the square of L.S.B and multiply it with M.S.B. and calculate the square of M.S.B. and multiply it with L.S.B.

Multiply the intermediate term with 11 (i.e. in decimal by 3).

Concatenate all the three parts which is the final result.

E.g.: Find $(11)^3$

Let A = 1 and B = 1

$(\text{M.S.B.})^3 = 1^3 = 1$ = L.H.S. of the final answer and

$(\text{L.S.B.})^3 = 1^3 = 1$ = R.H.S. of the final answer.

For finding two intermediate terms by using vertically and crosswise Sūtra:

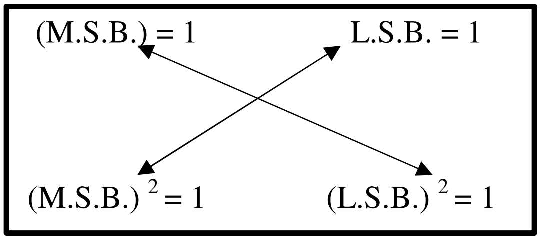


Figure: 6. 6

For the final result concatenate all the parts.

1 | 1 | 1 | 1

10 |10

10| 1 carry

1 1| 0 | 1 | 1

Final answer = $(11)^3 = 11011$

6.8 Applications and advantages of Vedic Sūtras in computer:

Vedic Mathematics **Sūtras** for multiplication, division and also for squaring and cubing operations are applied by many scholars belonging to areas like DSP, DFT, VLSI algorithm (with less power & greater speed), Algorithm to prepare new design of chip & RSA Encryption system.

FPGA implementation of block convolution, new multiplier architecture developed and embedded into OLA and OLS method by using Urdhva -Tiryagbhyam has improved efficiency in terms of area.

After comparing 4×4 bit Vedic multiplier on two different adder architectures concluded that the 4×4 bit Vedic multiplier with **carry look adder** is faster than 4×4 bit Vedic multiplier with ripple carry adder**.**

High speed reconfigurable FFT designed and implemented by using Urdhv-Tiryagbhyam algorithm boast rapid operations, lossless propogation and compact aesthetics.

After studying performance analysis of integrated multiplier architecture using **Ūrdhva-Tiryagbhyām & Nikhila□ Sūtra**, it is found that **Ūrdhva-Tiryagbhyām** works faster for small input and as the size of multiplication increases **Nikhila□** multiplier works better. Based on initial condition only one multiplier Sūtra performs the multiplication at any given time.

The speed of 4×4 bit multiplier with reversible logic by using Vedic Sūtra **Ūrdhva-Tiryagbhyām** increased and negligible power can be achieved by reversible logic gate. After the performance analysis among multipliers like **Ūrdhva-Tiryagbhyām, Nikhila□** and **Karatsuba** multiplier; for 8 × 8 bit Vedic Ūrdhva-Tiryagbhyām multiplier and for 16 × 16 bit Vedic Nikhila□ multiplier is faster. The 8 × 8 bit multiplier by Vertically & Crosswise Sūtra is more efficient than Array and Booth multiplier.

After comparison of various Vedic multiplication techniques like Ūrdhva-Tiryagbhyām, Nikhila□ & Ānurūpye□a it is found that Ūrdhva-Tiryagbhyām is most efficient Sūtra giving minimum delay for all types of numbers and reduces the area and speed up the computation.
Design of 64 bit Vedic multiplier by using Nikhila□ Sūtra has less delay and area compared to array and booth multiplier. The hardware implementation of 64 × 64 bit Vedic Nikhila□ multiplier using Barrel shifter contributes to adequate improvement of the speed in order to achieve high outturn than the conventional array multiplier.

A particular FFT which is designed by Vedic adder, subtractor and multiplier by using Vertically & Crosswise Sūtra provides feasible & rapid performance in the area of Discrete Fourier Transform and also rarely experiences any lag complaints.

RSA encryption and decryption algorithm by using Vertically & Crosswise Sūtra for multiplication and Dhwajaṅka property for division reduces time and reduces delay prominently as compared to the RSA implemented using traditional multipliers and division algorithm.

Implementation of fix and floating point division by Dhwajaṅka property of Vertically & Crosswise Sūtra is more efficient to achieve high speed cryptography algorithm in secure transaction.

After comparison results shows that the proposed Vedic square architecture by using Vedic Yāvadūnam Sūtra have both speed and area improvement over existing architecture.

Thus, Application of Vedic Sūtras improves the computational skill in a wide-ranging problems and is rigorously centered on lucidity & common-sense with great swiftness as well as accuracy.

SUMMARY

SWĀMĪ ŚRĪ BHĀRATĪ KṚṢṆA TĪRTHAJĪ MAHĀRĀJA , has given 16 **Sūtras** and 13 **Sub-Sūtras** which are written in a form of word formulae derived from Vedas. The researcher has tried to explain the meaning of **Vedic Sūtras** and **Sub-Sūtras** and applications by giving examples of each. Each formula deals with a different branch of Mathematics**.** **Sūtras** are very general and this is why they are so powerful and have such a wide range of applications. More research work in right direction needs to be done on each of these **Sūtras** summoning scholars from different fields to do research and decipher these. The world would surely find the right direction for exploring into advanced field.

7.1 SUMMARY AND CONCLUSION:

Chapter-1 Introduction and Mathematical Prerequisites:

In the 1st chapter of the thesis the researcher has explained about the founder of VM SWĀMĪ ŚRĪ BHĀRATĪ KṚṢṆA TĪRTHAJĪ MĀHĀRĀJA's life and his Journey of finding Vedic Mathematics **Sūtras** and **Sub-Sūtras** with their meaning and also discussed the link **of Vedic Sūtras** and **Sub-Sūtras** to the Arithmetic, Algebra and Calculus by giving proper examples solved by using conventional method as well as through Vedic Mathematics by applying related **Vedic Sūtras** & **Sub-Sūtras**. Then the scope and the importance of Vedic Mathematics is explained. After the comparison the method by using **Vedic Sūtras** and **Sub-Sūtras** with the conventional method, the researcher finds that the conventional methods are lengthier, complicated and time consuming than the method by using **Vedic Sūtras**. If one has the knowledge of the **Vedic Sūtras** and **Sub-Sūtras** and their applications to the particular problem then by using it he/she can solve the problem within a few seconds without writing so many steps and in an easy way. That is why Vedic Mathematics **Sūtras** and **Sub-Sūtras** can be very useful for the student's appearing for competitive exams. First chapter concludes with the outline of the thesis.

Chapter -2 Literature review:

Literature review is very important part of the thesis which inspired the researcher to select the topic. The current research was based on secondary data. Researcher has studied past research papers, research journals, books and other theses related to the current research topics. After studying the above materials, the researcher has found that majority of the research has done in the field of computer science by applying the **Vedic Sūtras** and **Sub-Sūtras** for binary subtraction, Multiplication, Division as well as in binary squaring and Cubing. Also research has been done in the comparison of conventional method with the Vedic mathematics by giving two different groups of students, problems at primary level; one group had to solve the problems by conventional method and another group had to solve them by Vedic Mathematics. Very few (hardly one or two) research papers were found on applications of **Vedic Sūtras** and **Sub-Sūtras** in Quadratic equations, Factorization as well as advance level topics like Determinant and Matrices, Derivatives, Integrations and Solutions of Ordinary and Partial Differential equations. Such a research gap which the researcher addresses as sufficient research work has not being done in the above topics. So it was very challenging task for the researcher. After reading the books related on Vedic Mathematics, the researcher has found two books more interesting, useful and informative in terms of the content to which give more weightage to Vedic Maths.

The book ‘Vedic Mathematics’ by **SWĀMĪ ŚRĪ BHĀRATĪ KṚṢṆA TĪRTĀJĪ MAHĀRĀJA, Śankarāchārya of Govardhana Matha, Puri,** Edited by Dr. V. S. Agrawala and also the book ‘Vertically and Crosswise’-Applications of Vedic Mathematics **Sūtras** by Pickles, Nicholas & Williams; both the books were found very useful as well as the study of Literature review found very helpful for the current research topics by the Researcher.

Chapter-3 Solution of Quadratic, Cubic and Quartic Equations:

In this chapter solutions of special types of Quadratic equations are explained by using **Vedic Sūtras**: **Śūnyam □ Sāmyasamuccaye, Vilokanam, Ānurūpye Śūnyamanyat**& **Parāvartya Yojayet**. Also factorization of Quadratic expression with one variable (x) and with two variables (x & y) by using **Ānurūpyeṇa** & **Ādyamadyenāntyamantyena** Sub-Sūtras are explained by giving proper examples for each. The easiest method of factorization of long homogeneous quadratic expression with three variables (x, y & z) by using two **Sub-Sūtras Ādyamadyenāntyamantyena** & **Lopanasthāpanābhyām** are explained by taking examples. First by using **Lopanasthapanābhyām** by eliminating any two variables (substituting zero to one variable at a time) out of three and retaining in the remaining two variables which can be factorized by using **Ādyamadyenāntyamantyena**. Thus, to get real factors of long expression fill the gaps of already obtained the above two set of factors. Similarly, factorization of long homogeneous quadratic expression with three variables and one independent term as well as with four variables are explained by giving proper examples for each by using **Ādyamadyenāntyamantyena, Lopanasthāpanābhyām Sub-Sūtras** in which three eliminations are done by eliminating two letters (substituting zero to two letters at a time) and factorized the obtained results. Thus with this obtained three sets of factors, by filling gaps real factors of long expression is obtained. The method of factorization by using **Vedic Sūtras** was found easier than the time consuming current method.

Chapter-4 Determinant and Matrices:

In this chapter by introducing Determinant and Matrices along with their inter relationship, evaluation of above parameters for second, third and fourth order has been explained by conventional method and then by **vertically & crosswise**. For third and fourth order by using conventional method after writing so many steps evaluation can be done which may be time consuming but by using **Vedic Sūtra** the same evaluation can be done within a single step without writing so many steps. The same procedure of finding determinant is applied to find Adjoint, Inverse & Rank of a matrix. Evaluation of special types of determinant extracting one, two or three elements by using **Vedic Sūtra by elimination and retention** has been explained with proper examples. Applications of determinant in solving simultaneous linear equations, finding non-zero solution of linear homogeneous equations and to check the consistency of linear algebraic equations are explained nicely by giving examples of each. Area of triangle can be found out by using determinant and also the condition of being three points collinear are explained. It is found that for the evaluation of second order determinant there was not much difference in writing steps by using conventional as well as **Vedic Sūtra** but the same for third and fourth order by using Vedic **Vertically & Crosswise Sūtra v**ery quickly compare to the conventional method.

Chapter -5 Derivative, Integration and Solving Ordinary and Partial Linear and Non-linear Differential Equations:

In this chapter Derivative and **Calana-Kalanābhyām**, Differential Calculus has been explained by using **Guṇaka-Samuccayaḥ Sūtra**. According to the **Sūtra**, for quadratic equation if we can factorize in two linear factors then by adding that two factors the first differential D_1 is obtained.

$$\text{i.e. } D_1 = \sum a \ \& \ D_2 = 2!$$

For cubic expression we can factorize into three linear factors then differentials are:

$D_1 = \sum ab$; $D_2 = 2!\sum a$ and $D_3 = 3!$

For bi-quadratic expression we can factorize them into four linear factor then,

$$D_1 = \sum abc;\ D_2 = 2!\sum ab,\ D_3 = 3!\sum a \text{ and } D_4 = 4!$$

By using the above **Sūtra**, the process of factorization is possible for the greatest degree in any variable.

Finding differentiation value to functions related to each other arithmetically both by conventional method and also by using **Vertically & Crosswise Sūtra** are explained with proper examples and line diagrams. Also 2nd order, 3rd order and 4th order derivatives of the given functions are explained by using binomial theorems and **Vertically & Crosswise Sūtra**. Problems of Integrations based on Integration of multiplication of two functions and based on partial fractions of the different types; in which denominator containing linear and unrepeated factors, denominator containing linear and repeated factors and denominator containing the quadratic equation which cannot be factorized further three types of problems solved by using conventional method as well as by using **Parāvartya Yojayet** & **Viloknam Sub-Sūtra**. Solution of ODE & PDE for linear as well as nonlinear type are explained with Taylor series and Maclaurin's theorem and by using **Vedic Sūtra 'By alternate elimination and retention'.**

Integral equations **'By alternate elimination and retention' Sūtra** and Digression: Differentiation of the ratio of the polynomials by using **Vertically & Crosswise Sūtra**. Ordinary and partial differential equations by using **'By alternate elimination and retention'** and nonlinear equations by using **Duplex** property of **Vertically & Crosswise Sūtra** and **'By alternate elimination and retention' Sūtras** are explained with examples.

Chapter-6 Applications of **Vedic Sūtras** in Computer Arithmetic:

An important part of CPU is Arithmetic Logic Unit (ALU) by which the actual mathematical operation like Addition, Multiplication, Division, Squaring and Cubing for each instruction is performed which are used frequently in hardware level for Fast Fourier Transform, Convolution & Filtering which are applications of Digital Signal Process. Additionally, squaring and cubing are used in Cryptography, Animation, and image processing etc. where the speed is a crucial performance characteristic.

The binary system is internal language of computer which has base 2 i.e. it uses only two bits 0 and 1. Conversion method of Binary to Decimal number, decimal to binary number and Two's complement of a binary number are explained with examples. In this chapter the researcher has explained arithmetic operations in binary number system by giving one example of each. Just like a decimal number system the arithmetic operations are performed in binary number system. Different types of multipliers and Division algorithms are used for binary operations. Therefore, it is necessary to have efficient multiplier in terms of great rapidity, less power consumption, accuracy and occupies less number of slices.

In this chapter applications of **Vedic Sūtras** in computer arithmetic binary operations by using related **Vedic Sūtras** with their algorithms are explained. **Vertically & Crosswise Sūtra** based multiplier is used for all types of multiplications by giving examples of 2 x 2 bit, 4 x 4 bit, 8 x 8 bit; for special type of binary multiplication **Sūtras** like **Nikhilaṁ, Ekanyūnena Pūrveṇa** & **Ānurūpyeṇa Sub-Sūtra** based multiplier are used. For all types of binary division **Dhwajaṇka**; to solve larger class of problem of binary division **Parāvartya Yojayet** and for the numbers

which are near to base 10 **Nikhilaṁ** based algorithm are applied. For binary squaring **Dwandwayoga** means **Duplex** property of **Vertically & Crosswise Sūtra** and **Yāvadūnam Sūtra** based squaring architecture; for binary cubing **Yāvadūnam Sūtra** and **Ānurūpyeṇa Sub-Sūtra** based cubic architecture are discussed by the researcher.
There are conventional multipliers like Booth, Wallace Tree, Combinational and Array & Sequential Multiplier. It was found that **Vedic multiplier**, for squaring and cubing architecture based on **Vedic Sūtras** is the most efficient multiplier which gives quick output, covers less space & intakes less power compared to conventional Multiplier.

CONCLUSION:

Vedic Mathematics **Sūtras** can be effectively used in basic as well as in higher mathematics.

In this age of cut throat competitions & the necessity for overall excellence, recruiters too have transformed their hiring process to account for these factors. Apitude tests often serve as initial screening to the highly sought positions in academia as well as industries. VM approach has time and again proved to offer a huge advantage in preparation for these events as well as for an individual's holistic development.

Intrinsic sūtra mathematics is useful in bringing back fun and interest of students in an abstruse subject like Mathematics. A systematic and developed study of Vedic Mathematics will be extremely useful for students and researchers.

The modern teaching of one way calculations are rigid and boring. Vedic Mathematics has general methods and also many methods that apply for special cases. These calculations can often be carried out independent of direction & orientation. Because of this flexibility, students can use their own approach which promotes creativity and intuition. In this rapidly changing world, flexibility and adaptability are absolutely essential for success.

By using **Vedic Sūtras** complicated and lengthy computations can be solved with greater accuracy and lesser time as compared to calculations based on conventional maths. VM also improves memory and creates greater mental alertness.

The most significant quality of Vedic Maths is its consistency. Because of this quality it creates stress-free and enjoyable environment. It inspires innovations. The beautiful coherence between arithmetic and algebra is clearly visible in the Vedic system.

Vedic algorithms based on **Ūrdhva-Tiryagbhyām Sūtra, Nikhilaṁ Sūtra** & **Ānurūpyeṇa Sub-Sūtra** etc. can be applied to design great rapidity Vedic Multipliers & reconfigurable Fast Fourier Transform (FFT) in DSP.

7.2 RECOMMENDATION:

Though Vedic maths, demands regular practice of simple problems, even a little exposure to the Vedic maths approach, coupled with some practice, clearly has immense added advantage as it develops a novel & instinctual approach to problem solving.

The impression that has been created over the years has put VM in a bad light reducing it to abstract hacks to special types of quantitative problems for which one has to memorize innumerable conditions, relations and operations. Large scale education programs to promote instrinsic logical approach will go a long way in popularizing what our ancestor knew way back & which is now being explored by the western world.

Inclusion of Vedic Mathematics in the curriculum right from the first standard will undoubtedly change the present attitude of students towards Mathematics as it allows for creative usages.

Researcher would like to highlight that we must try to develop the spirit of Vedic Mathematics in all upcoming computations and technologies by building upon the foundation work done in exploring the Contributions of Vedic Mathematics in Advance Calculus.

The propagation of Vedic Mathematics can be encouraged by conducting National Seminars and International Seminars at the Academic & Research level showcasing the applications of Vedic Mathematics in Computers.

Computers have entered in all areas of our life from banking, technology, transportation to education. Thus, there is a scope to apply powerful sutras in unexplored domains and scientific research for convenience and overall well being of humans.

By using the technology of Very Large Scale Integration i.e. VLSI with **Vedic Sūtra** designing and show casing Vedic DSP Chip is a very promising area for future applications of **Vedic Sūtras.**

7.3 FUTURE SCOPE IN CURRENT RESEARCH:

Going unrecognized & underappreciated for nearly a century, VM approach has risen in reputation among both intellectuals as well as schoolchildren primarily due to its flexibility. The freedom to approach a problem without the constraints of unconsciously abiding to a predefined set of abstract rules offers unparalleled opportunities in areas of scientific research & radical discovery in Information Technology.

When people are given an option to educate themselves through a self defined pace and are given access to resources that encourage development of their innate capabilities, they learn fast and retain more information and moreover get things done independently. This develops a sense of satisfaction in addition to making the previously boring quantitative studies exciting and enriching.

Further the **Ūrdhva-Tiryagbhyām** Method of solving linear simultaneously equation can be used for solving even 8 or 10 unknowns using the Pivoting techniques.

Previous researchers have limited their application domain mainly towards astronomy and classical mechanics. However with the rise in supercomputers, quantum computing and machine learning, the effective sūtra algorithms may work wonders as they perfectly fit the current needs of these trending technologies.

Further big data and analytics involve predictive modelling of infinite data generated globally in order to make business decisions. This online information helps companies better understand their consumer needs and help them meet their expectations while fulfilling the organiztions objective of productivity. VM could play an instrumental role in bringing these proposed technologies soon to reality.

VM creates a deeper understanding about the matrix of using computer generated translations and human language through the field of artificial intelligence. A study of Vedic maths is not only immediately useful but holds a lot of scope for further development and research.

7.4 LIMITATIONS:

This thesis has given more applications about the techniques of Vedic Mathematics-applied to Arithmetic, Algebra and Calculus and also to Computer Arithmetic. It does not discuss about the source of Vedic Mathematics.

The solutions of matrix problems by **Vertically & Crosswise Sūtra** are applicable to only the types which are based on determinants. The Vedic concepts may be very strong but the concepts cannot be generalized.

In Differential calculus, **Vedic Sūtras** are applied to the problems of derivatives and successive differentiation based on Polynomial functions only. However, the problems of derivatives of product of two functions and three functions can be solved by using **Vertically & Crosswise Sūtra** along with the help of binomial theorem and the knowledge of standard Derivative formulae of conventional mathematics. Integration by parts by using **Vertically & Crosswise Sūtra** and integration based on partial fraction by using **Parāvartya Yojayet Sūtra** can be faster in comparison to standard Integration methods. But for any other types except for the above two type's problems on Integration we have to rely on standard formula of Integration.

There are a very few research papers focusing solely on Contribution of Vedic Mathematics in Advance Calculus, hence the researcher did not have the privilege to have an access to a wide range of information.

Although there are various obstacles and limitations currently to employ the techniques of Vedic mathematics in applications involving lengthy and time consuming computations, these hindrances pose a challenge for future researchers to make this area of research interesting and encouraging

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1. Vaidya Sneha (2015),**“An effective Methods of Vedic Mathematics”,** “CENTUM”, Multidisciplinary Bi-Annual Research Journal, Special Issue-1,Vol-8, ISSN 2231-1475,pp: 349-356
2. Vaidya Sneha (2015), **“Vedic Maths: An aid to enhance Calculation Skill”,** International journal for Interdisciplinary Knowledge, Vol-1, Issue-3(2015-2016) ISSN: 2394-0018, pp: 89-93
3. Rastogi Prabha, Vaidya Sneha (2016), **“Applications of Vedic Mathematics in Determinants”,** International journal for Interdisciplinary Knowledge, ISSN: 2394-0018
4. Rastogi Prabha, Vaidya Sneha (2017), **“Applications of Vedic Sūtras in Factorization of Quadratic Expressions”,** International journal for Interdisciplinary Knowledge, Vol-1, and Issue 5(2017-2018), pp: 163-166
5. Vaidya Sneha (2018), **“Vedic Maths and Computer”,** Vidyawarta-Interdisciplinary Multilingual Refereed Journal, ISSN: 2319- 9318, pp: 15-18
6. Vaidya Sneha (2018), **“Algorithm of square architecture by using Vedic Mathematics”** International Journal of Renewable energy Exchange,ISSN:2321 - 1067
7. Vaidya Sneha (2019), **“Algorithm of cube architecture by using Vedic Mathematics”** International Journal for Interdisciplinary Knowledge, Vol- 1, Issue: 5(2018-19) pp: 95-99 ISSN: 2394-0018

NATIONAL AND INTERNATIONAL CONFERENCE ATTENDED:

1) Attended and published paper at **International** conference on Research Management held at shri Jagdishprasad Jhabarmal Tibrewala University, Vidyanagari, Chudela, Jhunjhunu, Rajasthan from October 30 to Nov.1, 2015.

2) Attended, presented and published paper at **National** Seminar on “Symbiotic Correlation of Mathematics and Information Technology Pan-Industry”held at Shailendra Education Society’s Arts, Commerce and Science College, Dahisar (E), Mumbai on 20th January, 2018

3) Attended, Presented and published paper at **International** conference on “Innovations in Multi-disciplinary approach” held at Shri M. D. Shah Mahila College, Malad (West) on 28thApril, 2018and Certified as **Best Research Paper.**

4) Attended **National** Conference on Yoga and Health, held at Shri Rajasthani Seva Sangh, J.B.Nagar, Andheri (East), Mumbai-59 on 21st June, 2018.